Additional Complexity Classes

We recall first some facts we mentioned in the past:

1. If C is a deterministic time or space complexity class, then $C = coC$. This means that C is closed under complements (complementations). In particulr, $coP = P$.
2. If C is a nondeterministic space complexity class, then $C = coC$. This means that C is closed under complements (complementations). In particular, $CoNPS = NPS$.
3. $NP = coNP$ iff there exists an $NP-complete$ problem whose complement is NP. In other words, $NP = coNP$ iff $NPC \cap coNP \neq \phi$.

Remarks:

1. PS is the same as $PSPACE$ and NPS is the same as $NPSPACE$.
2. Let C be a complexity class, then $L \in C$ iff $\overline{L} \in coC$.
3. IF A is a problem, then $A \in C$ iff A COMPLEMENT is coC.

Theorem 1. *L is NPC iff $\overline{L}$ is $CoNP-complete$.*

Proof. We have to prove that

1. $\overline{L} \in coNP$.
2. Every Q in $coNP$ is polynomial-time reducible to $\overline{L}$.

1. Since L is $NP-complete$, then L is NP and hence $\overline{L}$ is $coNP$.
2. Let $Q \in coNP$. Then $\overline{Q} \in NP$. But, $L \in NP-complete$ implies that there is a polynomial-time reduction R from $\overline{Q}$ to L. The same reduction is a reduction from Q to $\overline{L}$.

□

Theorem 2. *If $CoNP-complete \cap NP \neq \phi$, then $NP = coNP$.*

Proof. Because of symmetry, it suffices to prove that $coNP \subseteq NP$. Now let $Q \in CoNP - complete \cap NP$ and let $L \in coNP$. We will prove that $L \in NP$. But, $L \in coNP$ and $Q \in CoNP - complete \cap NP$ implies that there is a polynomial-time reduction R from L to Q. Also, $Q \in NP$ implies that there is a NDTM N that decides Q in polynomial-time. Now construct a NDTM N_1 that first computes the reduction $R(x)$, where x is the input string. Then pass $R(x)$ to N. This machine N_1 decides L in polynomial-time. Hence, $L \in NP$. $\square$

Open question: Is $NP = coNP$? The expected answer is no. So, people expect that NP is not closed under complementation. In fact, they expect that $NPC \cap coNP = \phi$. Also, they expect $coNPC \cap NP = \phi$.

Remarks:

1. **If $P = NP$, then $coNP = coP = P = NP$. Hence, if $P = NP$, then $NP = coNP$.**
2. **It is possible that $P \neq NP$ and $NP = coNP$.**
3. **There are problems in $NP \cap coNP$ that are not known to be in P. An example of such problems is PRIMES which will be defined later.**

Now we mention one more important problem.

Definition 3. The VALIDITY problem is the problem:

Given a BE in CNF, is it valid (*satisfiable by all truth assignments*)?

Nottice that VALIDITY is different than SAT COMPLEMENT. SAT COMPLEMENT (also known as USAT) is the problem: given a BE, is it unsatisfiable? In fact, SAT COMPLEMENT is the opposite of VALIDITY. Now since SAT is NPC, then SAT COMPLEMENT is $coNPC$, which means that VALIDITY is also $coNPC$.

Remark: VALIDITY and HAMILTON PATH COMPLEMENT are $coNP-$ *complete*, **which means that they are also** $coNP$ **and that every** $coNP$ **problem is reducible to each one of them in polynomial time.**

Now we define PRIMES

Definition 4. The PRIMES problem is the problem:

Given a positive integer N in binary, is it prime?

Theorem 5. *PRIMES is in* $NP \cap coNP$.

Remarks: The above theorem implies that PRIMES is probably not NPC, **because if it is** NPC, **then since it is also** $coNP$, **then by the one of the facts we mentioned at the beginning, we must have that** $NP = coNP$ **which is unlikely.**