

Boolean Expressions and NP-complete Problems

DEFINITION 1. A *boolean expression* is an expression built from:

- The binary operators $\vee$, $\wedge$.
- The unary operator $\sim$.
- Parantheses.

REMARK 0.1. Notice that the priority is for $\sim$, then $\wedge$, then $\vee$.

Note: We'll use 1 for true and 0 for false.

Notation: We'll use BE for *boolean expression*.

DEFINITION 2. Let E be a BE. A *truth assignment* for E is an assignment that assigns either true or false to each one of the variables of E .

Notation: We'll use TA for *truth assignment*.

Notation: If T is a TA for a boolean expression E , then we'll use $E(T)$ for the value of E corresponding to T .

DEFINITION 3. A truth assignment T *satisfies* a boolean expression E if $E(T) = 1$.

DEFINITION 4. A boolean expression E is said to be *satisfiable* if there exists at least one TA that satisfies E .

EXAMPLE 5. Let $E = x_1 \wedge \sim (x_2 \vee x_3)$. Then E is satisfiable. Consider the truth assignment T defined by: $T(x_1) = 1$, $T(x_2) = 0$, $T(x_3) = 0$. Then $E(T) = 1$.

EXAMPLE 6. $E = x_1 \wedge (\sim x_1 \vee x_2) \wedge (\sim x_2)$ is not satisfiable, because in order for a truth assignment T to satisfy E , we need $T(x_1) = 1$ and $T(x_2) = 0$. But, if so, then $\sim x_1 \vee x_2$ will be false. Hence, $E(T) = 0$.

DEFINITION 7. The *satisfiability problem*, denoted by SAT is the problem:

Given a BE , is it satisfiable?

THEOREM 8. *(Cook's Theorem) SAT is NP-complete.*

DEFINITION 9. A *literal* is a variable or a negated variable (e.g. x_1 , $\sim x_2$).

DEFINITION 10. A *clause* is one or more literals combined by $\vee$.

EXAMPLE 11. The following are clauses:

- x_1 .
- $x_1 \vee \sim x_2$.
- $x_1 \vee x_2 \vee x_3$.

DEFINITION 12. A *BE* is said to be in *conjunctive normal form (CNF)* if it consists of clauses combined by $\wedge$.

EXAMPLE 13. $x_1 \wedge (x_1 \vee x_2) \wedge (\sim x_1 \vee x_2 \vee x_3)$ is a *BE* in *CNF*.

DEFINITION 14. A *BE* is said to be in *k-conjunctive normal form (k-CNF)* if it is in *CNF* and if each clause has exactly k distinct literals.

EXAMPLE 15. $(x_1 \vee x_2 \vee \sim x_3) \wedge (x_1 \vee \sim x_2 \vee x_3)$ is in $3 - \text{CNF}$.

DEFINITION 16. The *CSAT* problem is the problem:

Given a *BE* in *CNF*, is it satisfiable?

DEFINITION 17. The $k - \text{SAT}$ problem is the problem:

Given a *BE* in $k - \text{CNF}$, is it satisfiable?

Note: Sometimes we'll write $k\text{SAT}$ instead of $k - \text{SAT}$.

THEOREM 18. *3SAT is NP-complete.*

DEFINITION 19. The *CLIQUE* problem is the problem:

Given an undirected graph G and a positive integer k , does G have a clique of size k .

Note: Sometimes the CLIQUE problem is stated as follows:

CLIQUE= $\{\langle G, k \rangle : G \text{ has a clique of size } k\}$.

THEOREM 20. *CLIQUE is NP-complete.*

PROOF. The proof was give in class. It was done in two steps.

- We proved that CLIQUE is *NP*.
- We reduced in polynomial time 3SAT to CLIQUE.

□

DEFINITION 21. The *VERTEX-COVER problem* is the problem:

Given an undirected graph G and a positive integer k , does G have a vertex cover of size k .

Note: Sometimes the VERTEX-COVER problem is stated as follows:

VERTEX-COVER= $\{\langle G, k \rangle : G \text{ has a vertex cover of size } k\}$.

THEOREM 22. *VERTEX-COVER is NP-complete.*

PROOF. The proof was give in class. It was done in two steps.

- We proved that VERTEX-COVER is *NP*.
- We reduced in polynomial time CLIQUE to VERTEX-COVER.

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