EXAM I

Name:

Instructions: Show your work and explain every step. You may not be given credit at all for incomplete solutions. Answers with no explanations will not be accepted. Do not write the numbers in the decimal form (keep them as fractions). Do not write on the back of the page.

(1) (6 points) Find the negation, contrapositive, and the converse of the following statement (make it clear which one is which).

If x is a prime number and $x > 2$, then x is odd.

(2) (6 points) Express the statement as a mathematical formula and find its negation.

There exist a real number y such that, for every real number x,
$$2x^2 + 1 > x^2 y.$$

(3) (10 points) Let $B = \{5, 4\}$, $C = [1, 4]$, $D = (2, 6]$, $E = \{4, a\}$. Find the following

(a) $\mathcal{P}(B)$.
(b) $B \times E$.
(c) $C \cup D$.
(d) $D - C$.
(e) $D \cap 2\mathbb{Z}$.

(4) (6 points) Prove or disprove

$$p \longrightarrow (q \vee r) \equiv (p \longrightarrow q) \vee (p \longrightarrow r).$$

(5) (4 points) What is the negation of $p \longleftrightarrow q$?

(6) (20 points) Decide whether the following are true or false

(a) Let A and B be sets. Then, $|A \cup B| = |A| + |B|$.

(b) Let A and B be sets. Then, $|A - B| = |A| - |A \cap B|$.

(c) $\overline{\overline{q} \longrightarrow \overline{p}} \equiv p \wedge \overline{q}$.

(d) $\{3\} \in \{1, 2, 3\}$.

(e) $\{3\} \subseteq \{1, 2, \{3\}\}$.

(f) $\phi \in \{2, 3, 4, \{5\}\}$.

(g) $\phi \subseteq \{2, 3, 4, \{5\}\}$.

(h) The sets $\{2, 3\}$, $\{4, 5\}$, and $\{3, 7\}$ are pairwise disjoint.

(i) The sets $\{a, b, c\}$, $\{d, e\}$, and $\{f, g\}$ form a partition of $\{a, b, c, d, e, f, g\}$.

(j) It's possible to have $\{4, 5, 6\}$ and $\{5, 7, 8\}$ as equivalence classes for an equivalence relation on $\{4, 5, 6, 7, 8, 9, 10, 11\}$.

(k) It's possible to have an equivalence relation that is both symmetric and antisymmetric.

(l) It's possible to have an equivalence relation that is both equivalence relation and a partial order.

(m) If 5 and 7 are elements of set A, $\mathcal{R}$ is an equivalnce relation on A, and the equivalence class of 5 is $\{5, 7, 8\}$, then the equivalence class of 7 is $\{5, 7, 8\}$.

(n) If 5, 6, and 7 are elements of set A and $\mathcal{R}$ is an equivalnce relation on A, then it's possible to have the equivalnce class of 5 to be $\{6, 7\}$.

(o) Let $A = \mathbb{Z}^+$, $\mathcal{R} = \{(x, y) \in A \times A \mid x|y)\}$. Then 3 and 5 are incomparable.

(p) If a and b are nonzero real numbers and $a|b$ and $b|a$, then $|b| = |a|$.

(q) If A and B are sets, then $A \oplus B = (A \cup B) - (A \cap B)$.

(r) $p \longrightarrow q \equiv \overline{q} \longrightarrow \overline{p}$.

(s) If A, B, and C, are any sets, then $\overline{A \cup (B \cap C)} = \overline{(A \cup B)} \cup \overline{(A \cup C)}$.

(7) (25 points) Prove or disprove the following:

(a) The set $S = \left\{\frac{1}{3n} \mid n \in \mathbb{Z}^+\right\}$ has a smallest element.

(b) If a and b are irrational, then $a+b$ is irrational.

(c) If a and c are in $\mathbb{R}$ with $a < c$, then $\exists b \in \mathbb{R}$ such that $a < b < c$.

(d) If n^2 is odd, then n is odd.

(e) If a and b are both irrational, then a^b is irrational.

(8) (10 points) Prove the following by mathematical induction:

$$1 + 3 + \cdots + (2n - 1) = n^2, \ \forall n \geq 1.$$

(9) (12 points)

(a) Let $A = \mathbb{Z}$, $\mathcal{R} = \{(x, y) \in A \times A \mid |2x - 3y| < 5)\}$. Prove by a counterexample that $\mathcal{R}$ is not reflexive.

(b) Let $A = \mathbb{Z}$, $\mathcal{R} = \{(x, y) \in A \times A \mid |2x - y| < 5)\}$. Prove by a counterexample that $\mathcal{R}$ is not transitive.

(c) Let $A = \mathbb{Z}$, $\mathcal{R} = \{(x, y) \in A \times A \mid x - y > 5)\}$. Prove that $\mathcal{R}$ is antisymmetric.

(d) Let $\mathcal{R} = \{(x, y) \in \mathbb{Z} \times \mathbb{Z} \mid x - y = 2k,\ k \in \mathbb{Z}\}$. Find $[7]$.

(e) Let $A = \mathbb{Z} - \{0\}$, $\mathcal{R} = \{(x, y) \in A \times A \mid x|y)\}$. Prove by a counterexample that $\mathcal{R}$ is not symmetric.

(f) Let $A = \mathbb{Z} - \{0\}$, $\mathcal{R} = \{(x, y) \in A \times A \mid xy > 0)\}$. Prove that $\mathcal{R}$ is transitive.