Linear Transformations

Notation: If $\vec{x} \in R^k$, then

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_k \end{bmatrix} \text{ or } \vec{x} = (x_1, x_2, \cdots, x_k).$$

Definition: A mapping (function) L from $\mathbb{R}^n$ to $\mathbb{R}^m$ (n and m are natural numbers) is called a *linear transformation* iff $L(a\vec{x} + b\vec{y}) = aL(\vec{x}) + bL(\vec{y})$, for all a and b in $\mathbb{R}$ and for all $\vec{x}$ and $\vec{y}$ in $\mathbb{R}^n$.

Theorem 1: Let A be an m by n matrix and let $L: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be $L(\vec{x}) = A\vec{x}$, for all $\vec{x} \in \mathbb{R}^n$. Then L is a linear transformation.

Proof: We have to prove that $L(a\vec{x} + b\vec{y}) = aL(\vec{x}) + bL(\vec{y})$, for all a and b in $\mathbb{R}$ and for all $\vec{x}$ and $\vec{y}$ in $\mathbb{R}^n$. But,

$$L(a\vec{x} + b\vec{y}) = A(a\vec{x} + b\vec{y}) = A(a\vec{x}) + A(b\vec{y}) = a(A\vec{x}) + b(A\vec{y}) = aL(\vec{x}) + bL(\vec{y}).$$

Theorem 2: If $L: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is a linear transformation, then there exists an m by n matrix A such that $L(\vec{x}) = A\vec{x}$ for all $\vec{x} \in \mathbb{R}^n$.

Remark: Combining the above two theorems we get the following theorem:

Theorem 3: $L: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is a linear transformation if and only if there exists an m by n matrix A such that $L(\vec{x}) = A\vec{x}$ for all $\vec{x} \in \mathbb{R}^n$.

Example: Let $L: \mathbb{R}^2 \longrightarrow \mathbb{R}$ be $L(\vec{x}) = x_1 + x_2$, for all $\vec{x} \in \mathbb{R}^2$. Then L is a linear transformation.

Proof: Done in class.

Example: Let $L: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be $L(\vec{x}) = (-x_2, x_1)$, for all $\vec{x} \in \mathbb{R}^2$. Then L is a linear transformation.

Proof: Done in class.

Example: Let t be a real number and let $L: \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be $L(\vec{x}) = t\vec{x}$, for all $\vec{x} \in \mathbb{R}^n$. Then L is a linear transformation.

Proof: We have to prove that $L(a\vec{x} + b\vec{y}) = aL(\vec{x}) + bL(\vec{y})$, for all a and b in $\mathbb{R}$ and for all $\vec{x}$ and $\vec{y}$ in $\mathbb{R}^n$. But,

$$L(a\vec{x} + b\vec{y}) = t(a\vec{x} + b\vec{y}) = t(a\vec{x}) + t(b\vec{y}) = a(t\vec{x}) + b(t\vec{y}) = aL(\vec{x}) + bL(\vec{y}).$$

Note: If you take $t = 1$ in the previous example, you get what's called the identity mapping (aka the identity transformation aka the identity operator).

Example: Let $L: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be $L(\vec{x}) = (1 + x_1, x_2)$, for all $\vec{x} \in \mathbb{R}^2$. Then L is not a linear transformation.

Proof:

$$L(a\vec{x} + b\vec{y}) = L\left(\begin{bmatrix} ax_1 + by_1 \\ ax_2 + by_2 \end{bmatrix}\right) = \begin{bmatrix} 1 + ax_1 + by_1 \\ ax_2 + by_2 \end{bmatrix}.$$

On the other hand,

$$aL(\vec{x}) + bL(\vec{y}) = a\begin{bmatrix} 1 + x_1 \\ x_2 \end{bmatrix} + b\begin{bmatrix} 1 + y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} a + b + ax_1 + by_1 \\ ax_2 + by_2 \end{bmatrix}.$$

Clearly $L(a\vec{x} + b\vec{y}) \neq aL(\vec{x}) + bL(\vec{y})$, for all a and b in $\mathbb{R}$ and for all $\vec{x}$ and $\vec{y}$ in $\mathbb{R}^2$. For example, take $a = b = 1$ and $x_1 = x_2 = y_1 = y_2 = 1$.

Exercises: Please do the following exercises. I'll post the solutions in the coming days.

Decide whether the following are linear transormations or not. Explain.

(1) $L: \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ defined by $L(\vec{x}) = (x_1, x_2, 1)$.

Solution: L is not a linear transformation, because

$$L\left(a\vec{x}+b\vec{y}\right)=L\left(\begin{bmatrix} ax_1+by_1 \\ ax_2+by_2 \end{bmatrix}\right)=\begin{bmatrix} ax_1+by_1 \\ ax_2+by_2 \\ 1 \end{bmatrix}.$$

On the other hand,

$$aL\left(\vec{x}\right)+bL\left(\vec{y}\right)=a\begin{bmatrix} x_1 \\ x_2 \\ 1 \end{bmatrix}+b\begin{bmatrix} y_1 \\ y_2 \\ 1 \end{bmatrix}=\begin{bmatrix} ax_1+by_1 \\ ax_2+by_2 \\ a+b \end{bmatrix}.$$

Clearly $L\left(a\vec{x}+b\vec{y}\right)\neq aL\left(\vec{x}\right)+bL\left(\vec{y}\right)$, for all a and b in $\mathbb{R}$ and for all $\vec{x}$ and $\vec{y}$ in $\mathbb{R}^2$. For example, take $a=b=1$ and $x_1=x_2=y_1=y_2=1$.

(2) $L:\ \mathbb{R}^2\longrightarrow\mathbb{R}$ defined by $L(\vec{x})=\sqrt{x_1^2+x_2^2}$.

Solution: L is not a linear transformation, because

$$L\left(a\vec{x}+b\vec{y}\right)=L\left(\begin{bmatrix} ax_1+by_1 \\ ax_2+by_2 \end{bmatrix}\right)=\sqrt{(ax_1+by_1)^2+(ax_2+by_2)^2}$$

On the other hand,

$$aL\left(\vec{x}\right)+bL\left(\vec{y}\right)=a\sqrt{x_1^2+x_2^2}+b\sqrt{y_1^2+y_2^2}.$$

Clearly $L\left(a\vec{x}+b\vec{y}\right)\neq aL\left(\vec{x}\right)+bL\left(\vec{y}\right)$, for all a and b in $\mathbb{R}$ and for all $\vec{x}$ and $\vec{y}$ in $\mathbb{R}^2$. For example, take $a=b=-1$ and $x_1=x_2=y_1=y_2=1$. Then, $L\left(a\vec{x}+b\vec{y}\right)=\sqrt{8}=2\sqrt{2}$, while $aL\left(\vec{x}\right)+bL\left(\vec{y}\right)=-2\sqrt{2}$

(3) $L:\ \mathbb{R}^2\longrightarrow\mathbb{R}^3$ defined by $L(\vec{x})=(x_2,x_1,x_1+x_2)$.

Solution: L is a linear transformation.

$$L\left(a\vec{x}+b\vec{y}\right)=L\left(\begin{bmatrix} ax_1+by_1 \\ ax_2+by_2 \end{bmatrix}\right)=\begin{bmatrix} ax_2+by_2 \\ ax_1+by_1 \\ a(x_1+x_2)+b(y_1+y_2) \end{bmatrix}.$$

$$aL(\vec{x}) + bL(\vec{y}) = a\begin{bmatrix} x_2 \\ x_1 \\ x_1 + x_2 \end{bmatrix} + b\begin{bmatrix} y_2 \\ y_1 \\ y_1 + y_2 \end{bmatrix} = \begin{bmatrix} ax_2 + by_2 \\ ax_1 + by_1 \\ a(x_1 + x_2) + b(y_1 + y_2) \end{bmatrix}.$$

Clearly $L(a\vec{x} + b\vec{y}) = aL(\vec{x}) + bL(\vec{y})$, for all a and b in $\mathbb{R}$ and for all $\vec{x}$ and $\vec{y}$ in $\mathbb{R}^2$.

(4) $L: \mathbb{R}^n \longrightarrow \mathbb{R}^n$ defined by $L(\vec{x}) = (0, -x_2, 0, \cdots, 0)$.

Solution: L is a linear transformation.

$$L(a\vec{x} + b\vec{y}) = L((ax_1 + by_1, ax_2 + by_2, \cdots, ax_n + by_n)) = (0, -ax_2 - by_2, 0, \cdots, 0).$$

$$aL(\vec{x}) + bL(\vec{y}) = a(0, -x_2, 0, \cdots, 0) + b(0, -y_2, 0, \cdots, 0) = (0, -ax_2 - by_2, 0, \cdots, 0).$$

Clearly $L(a\vec{x} + b\vec{y}) = aL(\vec{x}) + bL(\vec{y})$, for all a and b in $\mathbb{R}$ and for all $\vec{x}$ and $\vec{y}$ in $\mathbb{R}^n$.

(5) $L: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ defined by $L(\vec{x}) = (3, 5) + \vec{x}$.

Solution: L is not a linear transformation, because

$$L(a\vec{x} + b\vec{y}) = L\left(\begin{bmatrix} ax_1 + by_1 \\ ax_2 + by_2 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ 5 \end{bmatrix} + \begin{bmatrix} ax_1 + by_1 \\ ax_2 + by_2 \end{bmatrix} = \begin{bmatrix} 3 + ax_1 + by_1 \\ 5 + ax_2 + by_2 \end{bmatrix}.$$

On the other hand,

$$aL(\vec{x}) + bL(\vec{y}) = a\left(\begin{bmatrix} 3 \\ 5 \end{bmatrix} + \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) + b\left(\begin{bmatrix} 3 \\ 5 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}\right) = a\begin{bmatrix} 3 + x_1 \\ 5 + x_2 \end{bmatrix} + b\begin{bmatrix} 3 + y_1 \\ 5 + y_2 \end{bmatrix} = \begin{bmatrix} 3a + 3b + ax_1 + by_1 \\ 5a + 5b + ax_2 + by_2 \end{bmatrix}.$$

Clearly $L(a\vec{x} + b\vec{y}) \neq aL(\vec{x}) + bL(\vec{y})$, for all a and b in $\mathbb{R}$ and for all $\vec{x}$ and $\vec{y}$ in $\mathbb{R}^2$. For example, take $a = b = 1$ and $x_1 = x_2 = y_1 = y_2 = 1$.