

Exam I

- (1) (a) x is called a fixed point of function f if $f(x) = x$. Approximate the largest fixed point of f using Newton's method, where $f(x) = \frac{4}{x} + 5$. Make your initial approximate fixed point $x_0 = 4$. Do not count x_0 as one of the iterations. The termination conditions are

(i) The maximum number of iterations is 10.

(ii) $|f(x_k)| < 10^{-7}$.

To how many significant digits does your approximation approximate the true fixed point. What is the relative error?

- (b) Reduce the following system of equations to one equation in terms of x and find the root of the resulting equation that lies between 13 and 14 (i.e. begin your iterations with the interval $[13, 14]$) using the Regula Falsi Method (Method of the False Position). Terminate when you have 5 iterations. Don't forget to write the solution as an ordered pair (i.e. you need to give both x and y).

$$e^{(\frac{x}{10})} - y = 0.$$

$$2 \ln(y) - \cos(x) = 2.$$

- (2) Use Newton's method to approximate the root of $f(x) = (x - 4)^3$. The termination conditions are the same as those in the first part of the previous question. Then, use one of the acceleration techniques we learned to accelerate the convergence of the method. Use here the same termination conditions as before. In both cases, make your initial approximate root $x_0 = 0$. Do not count x_0 as one of the iterations.
- (3) Approximate the following integral using both the decomposite Trapezoid rule and the composite Simpson's Rule using $n = 4$. Find the upper bound of the error in each case using the formulas of the error given in class, then find the relative error

$$\int_0^4 (4x^3 + 3x^2) dx$$

- (4) Use the Lagrange interpolation polynomial to derive the formula

$$\int_0^1 f(x)dx \approx Af\left(\frac{1}{4}\right) + Bf\left(\frac{1}{2}\right) + Cf\left(\frac{3}{4}\right).$$

Note that you have to find the values of A , B , and C .

- (5) In the following question, recall that the Lagrange's interpolation of the points $(x_i, f(x_i))$, $i = 0, \dots, n$, is $\sum_{i=0}^n f(x_i)L_i(x)$.

- (a) Find the coefficient of x^n in the Lagrange's polynomial interpolation of (x_i, y_i) , $i = 0, 1, \dots, n$.
 (b) Prove that

$$f[x_0, x_1, \dots, x_n] = \sum_{i=0}^n f(x_i) \prod_{j=0, j \neq i}^n (x_i - x_j)^{-1}.$$

- (c) Now define the linear transformation T as follows: $Tf = \sum_{i=0}^n f(x_i)L_i(x)$; i.e. T maps the function f to the Lagrange's interpolation of f . Prove that $Tp = p$ for every polynomial p of degree less than or equal to n .

- (d) Prove that $\sum_{i=0}^n L_i(x) = 1$, $\forall x$.

- (e) Show that if u is a function that interpolates f at x_i , $i = 0, \dots, n-1$, and if v is a function that interpolates f at x_i , $i = 1, \dots, n$, then the function

$$\left(\frac{(x_n - x)u(x) + (x - x_0)v(x)}{x_n - x_0} \right)$$

interpolates f at x_i , $i = 0, \dots, n$.

- (6) (a) Let S be the natural cubic spline that interpolates $(-1, 0)$, $(0, 2)$, and $(1, 4)$. Which properties of S does the following function g possess?

$$g(x) = \begin{cases} (x+1) + (x+1)^3 & x \in [-1, 0] \\ 4 + (x-1) + (x-1)^3 & x \in [0, 1] \end{cases}$$

- (b) Find a natural cubic spline that interpolates $(0, 1)$, $(1, 1)$, $(2, 0)$, $(3, 10)$.
 (7) (a) 4(b) Page 276 of the textbook.
 (b) 8(b) Page 261 of the textbook.

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