Exercises on Nullspace and on Functions

Definition: $C^{(n-1)}[a,b]$ is the vector space of all functions f defined on $[a,b]$ such that f and its first $n-1$ derivatives are continuous on $[a,b]$.

Definition: Let $f_1,\ f_2,\cdots,f_n$ be functions in $C^{(n-1)}[a,b]$. The **Wronskian** of $f_1,\ f_2,\cdots,f_n$, denoted $W[f_1,f_2,\cdots,f_n]$ is the function defined as follows:

$$W[f_1,\ f_2,\cdots,f_n](x)=\begin{vmatrix} f_1(x) & f_2(x) & \cdots & f_n(x) \\ f_1'(x) & f_2'(x) & \cdots & f_n'(x) \\ \vdots & \vdots & \vdots & \vdots \\ f_1^{(n-1)}(x) & f_2^{(n-1)}(x) & \cdots & f_n^{(n-1)}(x) \end{vmatrix}$$

Theorem Let $f_1,\ f_2,\cdots,f_n$ be elements in $C^{(n-1)}[a,b]$. If there exists a point $x_0\in[a,b]$ such that $W[f_1,\ f_2,\cdots,f_n](x_0)\neq 0$, then $f_1,\ f_2,\cdots,f_n$ are linearly independent.

Remark: If $f_1,\ f_2,\cdots,f_n$ are linearly independent in $C^{(n-1)}[a,b]$, then they are linearly independent in $C[a,b]$.

Exercises:

(1) Show that e^x and e^{-x} are linearly independent in $C(-\infty,\infty)$.
(2) Show that x^2 and $x|x|$ are linearly independent in $C[-1,1]$.
(3) Show that $1,t,t^2,t^3,t^4$ are linearly independent in P_5.
(4) Show that the following are linearly independent in $C[-1,1]$.
   (a) $\cos(\pi x)$, $\sin(\pi x)$.
   (b) $x^{3/2}$, $x^{5/2}$.
   (c) 1, $2cosh(x)$, $2sinh(x)$.
   (d) e^x, e^{-x}, e^{2x}.
(5) Determine if the following are subspaces of $C[-1,1]$.
   (a) The set of all functions in $C[-1,1]$ such that $f(-1)=f(1)$.
   (b) The set of all odd functions on $C[-1,1]$.
   (c) The set of continuous nondecreasing functions on $C[-1,1]$.

(d) The set of functions in $C[-1,1]$ such that $f(-1)=0$ and $f(1)=0$.

(e) The set of functions in $C[-1,1]$ such that $f(-1)=0$ or $f(1)=0$.

(6) Determine the nullspace of the following matrices

(7) $A = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{bmatrix}.$

(8) $A = \begin{bmatrix} 1 & 2 & -3 & -1 \\ -2 & -4 & 6 & 3 \end{bmatrix}.$

(9) $A = \begin{bmatrix} 1 & 3 & -4 \\ 2 & -1 & -1 \\ -1 & -3 & 4 \end{bmatrix}.$

(10) $A = \begin{bmatrix} 1 & 1 & -1 & 2 \\ 2 & 2 & -3 & 1 \\ -1 & -1 & 0 & -5 \end{bmatrix}.$

(11) $A = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}.$