Basis

Definition: A susbset S of a vector space V is said to be a **basis** (plural bases) for V if S spans V and S is linearly independent.

Definition: The **dimension** of a nonzero vector space V, denoted dim V, is the number of vectors in a basis for V. The vector space $\{0\}$ has dimension zero. A vector space of dimension n is called n-dimensional.

Definition: A vector space V is called **finite dimensional** if there is a finite subset of V that is a basis for V. If there is no such subset, we call V **infinite dimensional**.

(1) Let V be a vector space of dimension $n > 0$. Then

(a) A basis cannot contain repeated vectors or the zero vector.

(b) $\{e_1, e_2, \cdots, e_n\}$ is a basis for $\mathbb{R}^n$, and $\{1, t, t^2, \cdots, t^n\}$ is a basis for P_n. Those bases are called standard/natural bases. Thus, dim $R^n = n$ and dim $P_n = n + 1$.

(c) The matrices A_{ij}, $i = 1, \cdots, n$, $j = 1, \cdots, m$, where A_{ij} is the matrix whose ij^{th} entry is 1 and the remaining entries are zeros form a basis for M_{mn}. This basis is call the standard/natural basis. Thus, dim $M_{mn} = mn$.

(d) Any set of n linearly independent vectors spans V, and hence, it's a basis for V.

(e) Any n vectors that span V are linearly independent, and hence, it's a basis for V.

(f) If $m > n$, then any collection of m vectors in V is linearly dependent.

(g) No set of less than n vectors can span V.

(h) If $S = \{v_1, v_2, \cdots, v_n\}$ is a basis for V and $T = \{w_1, w_2, \cdots, w_k\}$ is linearly independent, then $k \leq n$.

(i) If $S = \{v_1, v_2, \cdots, v_n\}$ is a basis for V and $T = \{w_1, w_2, \cdots, w_k\}$ are bases for V, then $k = n$.

(j) Thus, given a set S of m vectors in V. If $m > n$, then S is linearly dependent and hence not a basis for V. If $m < n$, then S does not span

V and hence not a basis for V. If $m = n$, then it suffices to check if S is linearly independent. If yes, then it's a basis for V. If not, then it is not a basis.

(k) Any subset of less than n linearly independent vectors can be extended to form a basis for V. Here is how to do that if the vector space is $\mathbb{R}^n$ or P_n: Suppose that you want to extend $\{v_1, \cdots, v_k\}$ where $k < n$ to a basis for V and let $b_1,\ b_2, \cdots, b_n$ be a basis for V. Form the matrix in which the first k columns are $v_1, \cdots, v_k$ and make the remaining columns $b_1,\ b_2, \cdots, b_n$. Now perform Gauss-Jordan elimination (or Gaussian elimination) on the matrix to reduce it to reduced rwo echelon form (or row echelon form), then the vectors corresponding to the columns with the leading 1's are the basis you're looking for.

(l) If S is a basis for a vector space V, then every vector v in V can be written uniquely as a linear combination of the vectors of S.

(m) Every nonzero vector space has infinitely many bases, because, for example, if $\{v_1, v_2, \cdots, v_n\}$ is a basis for a vector space V and c is a nonzero number, then $\{cv_1, v_2, \cdots, v_n\}$ is also a basis for V.

(2) Let $S = \{v_1, \cdots, v_m\}$ be a set in a vector space V. Then

(a) If one of the vectors in S, say v_j, is a linear combination of the remaining vectors in S, then span $S - \{v_j\} =$ span S.

(b) If Span $S \neq \{0\}$. Then some subset of S is a basis for span S. See how in the attached sheet.

Remarks:

(1) A subspace has to contain zero. Thus, any subspace of $\mathbb{R}^2$ must pass through the origin. Trivial subspaces of $\mathbb{R}^2$ are $\{0\}$ (dimension 0) and itself (dimension 2). One-dimensiona subspaces of $\mathbb{R}^2$ are lines through the origin.

(2) The subspaces of $\mathbb{R}^3$ are itself (dimension 3), $\{0\}$ (dimension 0), all lines through the origin (dimension 1), and all planes through the origin (dimension 2).

Review

(1) To check if $S = \{v_1, v_2, \cdots, v_n\}$ spans $V = \mathbb{R}^m$, you do the following:

(a) If $n < m$, then S does not span $\mathbb{R}^m$. If you were not asked to find a vector in V that does not belong to span S, you can stop here. Otherwise, you have to do the procedure in (b) below:

(b) If $n \geq m$, then you form the matrix A whose columns are the vectors of S, and then you solve the system $Ac = b$ by Gaussian elimination or Gauss-Jordan elimination, where b is an arbitrary vector in $\mathbb{R}^m$. That means the elements of b should be symbols. In other words, you have to form the augmented matrix $[v_1\ v_2\ \cdots\ v_n\ b]$, and then transform it to row echelon form or reduced row echelon form and solve the new system you get. **If** the system is consistent (i.e. has a solution) no matter what the vector b is, then S spans $\mathbb{R}^m$. **If** the system is inconsistent (i.e. has no solution) for some choices of vector b, then S does not span $\mathbb{R}^m$. In this case, the row echelon form or the reduced row echelon form of the augmented matrix must contain a row whose entries are all zeros except the last entry which could be nonzero for some values of vector b. **If** the system is inconsistent and you were asked to find a vector in V that does not belong to span S, then find a vector b that makes the last entry of the row, whose elements are all zeros except the last entry, nonzero. This vector does not belong to span S. **Now if** S spans $\mathbb{R}^m$ and you were asked to write one of the vectors of $\mathbb{R}^m$ (say v) as a linear combination of the vectors in S, then you need to solve the system $Ac = v$. When you do that you'll get the unkowns which are the elements of c (let's call these elements c_1, c_2, $\cdots$, c_n). After that you write $v = c_1v_1 + c_2v_2 + \cdots + c_nv_n$.

Remark: *Note that it's always sufficient to do the step above as we did in Section 2.3 no matter what the relationship between n and m is.*

(2) To check if $S = \{v_1, v_2, \cdots, v_n\}$ is linearly independent in $V = \mathbb{R}^m$, you do the following:

(a) If $n > m$, then S is linearly dependent. If you were not asked to write one of the vectors in S as a linear combination of the other vectors in S, you can stop here. Otherwise, you'll have to do the procedure in (b) (ii) below.

(b) (i) If $n = m$, then find $\det([v_1\ v_2\ \cdots\ v_n])$. If it is not zero, then S is linearly independent. If it's zero, then S is linearly dependent. In this case, you were not asked to write one of the vectors in S as a linear combination of the other vectors in S, you can stop here. Otherwise, you'll have to do the procedure in (ii) below.

(ii) If $n < m$, then you form the matrix A whose columns are the vectors of S, and then you solve the system $Ac = 0$ by Gaussian elimination or Gauss-Jordan elimination. In other words, you have to form the augmented matrix $[v_1\ v_2\ \cdots\ v_n\ 0]$, and then transform it to row echelon form or reduced row echelon form and solve the new system you get. (Note that there is no need to write the zero column at the end.) **If** the system has only the trivial solution, then S is linearly independent. **If** the system has a nontrivial solution (in this case it must have infinitely many solutions which means you get arbitary elements in the solution), then S is linearly dependent. **Now if** S is linearly dependent and you were asked to write one of the vectors in S as a linear combination of the other vectors in S, then you find one of the nontrivial solutions (say it's c, where the elements of c are $c_1,\ c_2, \cdots, c_n$). Now since c is a solution, then we must have $c_1v_1 + c_2v_2 + \cdots + c_nv_n = 0$. Now solve for one of the $v_i's$ (i.e. just leave one of the v_i's on the left hand side and move everything else to the other side).

Remark: *Note that it's always sufficient to do the step above as we did in Section 2.3 no matter what the relationship between n and m is.*

(3) To check if $S = \{v_1, v_2, \cdots, v_n\}$ is a basis for $V = \mathbb{R}^m$, you do the following:

(a) If $n \neq m$, then S is not a basis (if $n > m$, then S is linearly dependent and also it may not span $\mathbb{R}^m$; if $n < m$, then S does not span $\mathbb{R}^m$ and also it may be linearly dependent).

(b) If $m = n$, then it suffices to check if S is linearly independent using the method described above or by checking the determinant of $[v_1\ v_2\ \cdots\ v_n]$ (nonzero determinant implies linearly independent and zero determinant implies linearly dependent). If S is linearly independent, then it's a basis. If S is not linearly independent, then it is not a basis. Note that if S contains the zero vector, repeated vectors, or a vector that is a multiple of another, you can say right away S is linearly dependent and, hence, not a basis. Thus, there is no need in this case to do any further work unless you're asked to.

Now if S is a basis for $\mathbb{R}^m$ and you were asked to write a given vector v in $\mathbb{R}^m$ as a linear combination of the vectors in S, then you need to solve the system $Ac = v$. When you do that you'll get the unkowns which are the elements of c (let's call these elements $c_1,\ c_2,\ \cdots,\ c_n$). After that you write $v = c_1v_1 + c_2v_2 + \cdots + c_nv_n$.

(4) To find a basis from $S = \{v_1, v_2, \cdots, v_n\}$ for span S, you do the following (this procedure applies only if S is a subset of $\mathbb{R}^m$): You form the matrix $[v_1\ v_2\ \cdots\ v_n]$, and then transform it to row echelon form **or** reduced row echelon form. Then take the vectors of S corresponding to the columns (of the row echolon form or the reduced row echelon form) that contain the leading 1's. **Note** that the order of the vectors of S determines which basis for span S is obtained. I.e. if you change the order of the vectors in S, you may get a different basis for span S.

If S is not a subset of $\mathbb{R}^m$ or P_m, then find the row echelon form or the reduced row echelon form of $[v_1\ v_2\ \cdots\ v_n\ 0]$. If the only solution is the trivial solution, then S is linealy independent, and hence, it's a basis for its span. If there is a nontrivial solution, then S is linearly dependent, and hence one of the vectors in S can be written in terms of the remaining vectors of S. Now delete this vector from S and repeat the procedure on the new S. Continue doing that until you get a linearly independent set (whose span is equal to the span of the original S).

(5) To extend a linearly independent set $S = \{v_1, v_2, \cdots, v_n\}$ to form a basis for $\mathbb{R}^m$ where $n < m$, you do the following: You form the matrix $[v_1\ v_2\ \cdots\ v_n\ e_1\ e_2\ \cdots\ e_m]$, and then transform it to row echelon form **or** reduced row echelon form. Then take the vectors corresponding to the columns (of the row echolon form or the reduced row echelon form) that contain the leading 1's. Note that e_i is the ith column of the $m \times m$ identity matrix. I.e. the $e_i's$ are the standard basis for $\mathbb{R}^m$. Note also that the basis you get here may contain vectors from S and vectors from the standard basis of $\mathbb{R}^m$.

(6) To do any of the above items for P_m, you transform the question to a question similar to the above by taking the coefficients of each polynomial you're given and writing that as a vector.