

Span and Linear Independence

Definition: The vectors $v_1, v_2, \dots, v_n$ in a vector space V are said to span V if every vector in V is a linear combination of $v_1, v_2, \dots, v_n$. If $S = \{v_1, v_2, \dots, v_n\}$, we say S spans V , or V is spanned by S , or $\text{Span } S = V$.

Remarks:

- (1) To check if $v_1, v_2, \dots, v_n$ span a vector space V , do the following
 - (a) Choose an arbitrary vector v in V .
 - (b) If v is a linear combination of the given vectors, then the given vectors span V . If not, then they do not span V .
- (2) The set $S = \{e_1, e_2, \dots, e_n\}$ spans $\mathbb{R}^n$.
- (3) The set $S = \{1, t, t^2, \dots, t^n\}$ spans P_n .

Definition: The vectors $v_1, v_2, \dots, v_n$ in a vector space V are said to be linearly dependent if there exists constants $c_1, c_2, \dots, c_n$ not all zero such that $c_1v_1 + c_2v_2 + \dots + c_nv_n = 0$. If $v_1, v_2, \dots, v_n$ are not linearly dependent, then they are called linearly independent. Thus, $v_1, v_2, \dots, v_n$ are linearly independent if and only if whenever $c_1v_1 + c_2v_2 + \dots + c_nv_n = 0$, then $c_1 = c_2 = \dots = c_n = 0$. If $S = \{v_1, v_2, \dots, v_n\}$ and $v_1, v_2, \dots, v_n$ are linearly dependent, we say S is linearly dependent and if $v_1, v_2, \dots, v_n$ are linearly independent, we say S is linearly independent.

Remark: To check if $v_1, v_2, \dots, v_n$ are linearly independent/dependent, do the following

- (1) Form the equation $c_1v_1 + c_2v_2 + \dots + c_nv_n = 0$.
- (2) If the homogeneous system obtained has only the trivial solution, then the vectors are linearly independent; if it has a nontrivial solution, then the vectors are linearly dependent.

Definition: The nullspace of an $m \times n$ matrix A (or of $Ax = 0$), denoted $\text{Nul } A$, is the set of all solutions of $Ax = 0$. The nullspace of A is also called the solution space of the system $Ax = 0$.

Theorem:

- (1) The nullspace of an $m \times n$ matrix A is a subspace of R^n . Note that the nullspace of A is nonempty because it contains the trivial solution.
- (2) $\text{Nul } A = \{0\}$ iff $Ax = 0$ has only the trivial solution.
- (3) If the number of equations in a homogeneous linear system is less than the number of unknowns/variables, then the system has infinitely many solutions (which also implies it has a nontrivial solution).
- (4) If A is an $n \times n$ singular matrix, then $Ax = 0$ has infinitely many solutions; and if it's nonsingular, then $Ax = 0$ has only the trivial solution.
- (5) If a finite set S of vectors contains the zero vector, then S is linearly dependent.
- (6) Two vectors are linearly dependent iff one of them is a multiple of the other.
- (7) The nonzero vectors $v_1, v_2, \dots, v_n$ in a vector space V are linearly dependent iff one of the vectors, v_j , $j \geq 2$, is a linear combination of the preceding vectors $v_1, v_2, \dots, v_{j-1}$.
- (8) A finite set S of vectors in a vector space V is linearly dependent iff one of the vectors in S is a linear combination of all the other vectors in S .
- (9) Let A and B be finite subsets of a vector space V , and let $A \subseteq B$. Then,
 - (a) If A is linearly dependent, then so is B .
 - (b) If B is linearly independent, then so is A .
 - (c) If A spans V , then so is B .
- (10) To find a set that spans the nullspace of matrix A , solve the system $Ax = 0$. Then factor out all arbitrary variables you get. For example, if there are two arbitrary variables, say s and t , then the general solution you'll get will be something like this: $sa + tb$, where a and b are vectors free from all variables (i.e. they contain no s and no t). In this case, the set $\{a, b\}$ spans the nullspace of A .