

Real Vector Spaces and Subspaces

Definition: A real vector space is a set V of elements (called vectors) together with two operations $\oplus$ (called vector addition) and $\odot$ (called scalar multiplication) satisfying the following 10 properties (axioms):

- (α) If u and v are elements of V , then $u \oplus v$ is in V .
 - (a) $u \oplus v = v \oplus u$, for all u and v in V .
 - (b) $u \oplus (v \oplus w) = (u \oplus v) \oplus w$, for all u, v , and w in V .
 - (c) There is an element 0 in V such that $u \oplus 0 = 0 \oplus u = u$, for all u in V .
 - (d) For each u in V , there is an element $-u$ in V such that $u \oplus -u = 0$.
- (β) If u is an element of V and $c \in \mathbb{R}$, then $c \odot u$ is in V .
 - (e) $c \odot (u \oplus v) = c \odot u \oplus c \odot v$, for all $c \in \mathbb{R}$ and all u and v in V .
 - (f) $(c + d) \odot u = c \odot u \oplus d \odot u$, for all c and d in $\mathbb{R}$ and all u in V .
 - (g) $c \odot (d \odot u) = (cd) \odot u$, for all c and d in $\mathbb{R}$ and all u in V .
 - (h) $1 \odot u = u$, for all u in V .

Remarks:

- (1) If u is a vector in a vector space, then u is written as $\mathbf{u}$ or $\vec{u}$.
- (2) If the real numbers (called scalars) in the axioms above are complex, the vector space in that case is called a complex vector space.
- (3) The vector $-\mathbf{u}$ is called the negative of $\mathbf{u}$ and the vector $\mathbf{0}$ is called the zero vector.
- (4) $-\mathbf{u}$ and $\mathbf{0}$ are unique.
- (5) We will write sometimes $u \oplus v$ as $u + v$ and $c \odot u$ as cu .
- (6) Vectors of $\mathbb{R}^n$ are sometimes written as column vectors and sometimes as $(x_1, x_2, \dots, x_n)$.

Theorem: Let V be a vector space

- (1) $0 \odot u = 0$, for every u in V .
- (2) $c \odot 0 = 0$, for every real number c .

- (3) If $c \odot u = 0$, then either $c = 0$ or $u = 0$ (c here is a real number and $u \in V$).
- (4) $(-1) \odot u = -u$, for all $u \in V$.

Examples of Vector Spaces:

- (1) $\mathbb{R}$ with addition of real numbers and multiplication of real numbers.
- (2) $\mathbb{R}^n$, $\forall n \in \mathbb{Z}^+$ with addition of vectors and scalar multiplication.
- (3) M_{mn} : The set of all $m \times n$ matrices with matrix addition and scalar multiplication.
- (4) $F[a, b]$: The set of all real-valued functions defined on $[a, b]$, where $\oplus$ and $\odot$ are defined as follows: $(f \oplus g)(t) = f(t) + g(t)$, $(c \odot f)(t) = cf(t)$.
- (5) $F(-\infty, \infty)$: The set of all real-valued functions defined on $(-\infty, \infty)$.
- (6) $C[a, b]$: The set of all continuous real-valued functions defined on $[a, b]$, where $\oplus$ and $\odot$ are defined as follows: $(f \oplus g)(t) = f(t) + g(t)$, $(c \odot f)(t) = cf(t)$.
- (7) $C(-\infty, \infty)$: The set of all continuous real-valued functions defined on $(-\infty, \infty)$.
- (8) P_n : The set of all polynomials of degree $\leq n$ (that includes the zero polynomial), where $\oplus$ and $\odot$ are defined as follows: For $p(t) = a_0 + a_1t + a_2t^2 + \cdots + a_nt^n$ and $q(t) = b_0 + b_1t + b_2t^2 + \cdots + b_nt^n$ in P_n and $c \in \mathbb{R}$, $p(t) \oplus q(t) = (a_0 + b_0) + (a_1 + b_1)t + (a_2 + b_2)t^2 + \cdots + (a_n + b_n)t^n$, and $c \odot p(t) = ca_0 + ca_1t + ca_2t^2 + \cdots + ca_nt^n$.
- (9) P : The set of all polynomials (that includes the zero polynomial).

Remark: We'll be dealing only with real vector spaces (although the field $\mathbb{R}$ can be replaced by any field such as the field $\mathbb{C}$ of complex numbers). Thus, we'll assume all vector spaces below are real vector spaces (i.e. they are vector spaces over $\mathbb{R}$).

Definition: Let V be a vector space and let W be a nonempty subset of V . If W is a vector space with respect to the operations in V , then W is called a subspace of V .

Definition: Let V be a vector space and let W be a nonempty subset of V . Then W is a subspace of V if and only if

- (α) For every u and v in W , $u \oplus v$ is in W .

(β) For every u in W and every real number c , $c \odot u$ is in W .

Remarks:

(1) (α) and (β) in the definition above can be replaced by

For every u and v in W , and every c and d in $\mathbb{R}$, $c \odot u \oplus d \odot v$ is in W .

(2) If the word "nonempty" in the previous example is deleted, then we should add the following item to (α) and (β) above:

(γ) $0 \in W$.

(3) If V is a vector space, then V and $\{0\}$ are subspaces of V . $\{0\}$ is called the zero subspace.

(4) Every vector space must contain the zero vector.

(5) If a subset W of a vector space V does not contain the zero vector, then W is not a subspace of V . But, if a subset W of a vector space does contain the zero vector that does not necessarily means W is a subspace of V .

(6) Let A be an $m \times n$ matrix and let W be the set of all solutions to $Ax = 0$. Then, W is a subspace of $\mathbb{R}^n$ and it's called the *nullspace* of A or the *solution space* of the system $Ax = 0$.

(7) P_n is a subspace of P_{n+1} and P_n is a subspace of P .

(8) To check if a subset W of a vector space V is a subspace of V , you need first to check that W is nonempty. If W is nonempty, then check closure (with respect to W) of $\odot$ and $\oplus$. If W is not closed under $\odot$ or under $\oplus$, then it is not a subspace.

Definition: Let $v_1, v_2, \dots, v_k$ be vectors in a vector space V . A vector v in V is called a *linear combination* of $v_1, v_2, \dots, v_k$ if there exist real numbers $c_1, c_2, \dots, c_n$, such that

$$v = c_1 v_1 + c_2 v_2 + \dots + c_k v_k.$$

Remark: To check if a vector v is a linear combination of $v_1, v_2, \dots, v_k$, check if the linear system $c_1 v_1 + c_2 v_2 + \dots + c_k v_k = v$ has a solution (note that the unknowns here are $c_1, c_2, \dots, c_k$). If yes, then v is a linear combination of the given vectors. If not, then not.

Definition: Let $S = \{v_1, v_2, \dots, v_k\}$ be a set of vectors in a vector space V . Then the set of all vectors in V that are linear combinations of the vectors in S is called span S or $\text{span}\{v_1, v_2, \dots, v_k\}$.

Theorem: Let $S = \{v_1, v_2, \dots, v_k\}$ be a set of vectors in a vector space V . Then span S is a subspace of V .

Exercises:

- (1) Determine whether the following sets V are closed under $\oplus$ and $\odot$
 - (a) V is the set of all ordered pairs of real numbers (x, y) , where $x > 0$ and $y > 0$; $(x, y) \oplus (u, v) = (x + u, y + v)$, $c \odot (x, y) = (cx, cy)$.
 - (b) V is the set of all ordered triples of real numbers of the form $(0, x, y)$; $(0, x, y) \oplus (0, u, v) = (0, x + u, y + v)$, $c \odot (0, x, y) = (0, 0, cz)$.
 - (c) V is the set of all polynomials of the form $at^2 + bt + c$, where a, b , and c , are real numbers with $b = a + 1$; $(a_1t^2 + b_1t + c_1) \oplus (a_2t^2 + b_2t + c_2) = (a_1 + a_2)t^2 + (b_1 + b_2)t + (c_1 + c_2)$, $r \odot (at^2 + bt + c) = rat^2 + rbt + rc$.
- (2) Determine whether the given set V with the given operations is a vector space:
 - (a) V is the set of all ordered pairs of real numbers (x, y) ; $(x, y) \oplus (u, v) = (x + u, y + v)$, $c \odot (x, y) = (0, 0)$.
 - (b) V is the set of all real numbers; $u \oplus v = 2u - v$, $c \odot u = cu$.
 - (c) V is the set of all positive real numbers; $u \oplus v = uv$, $c \odot u = u^c$.
- (3) Show that if $u \neq 0$ and $a \odot u = b \odot u$, then $a = b$.
- (4) Which of the following subsets of $\mathbb{R}^3$ are subspaces of $\mathbb{R}^3$? The set W of all vectors of the form
 - (a) (a, b, c) , where $a = c = 0$.
 - (b) (a, b, c) , where $a = -c$.
 - (c) (a, b, c) , where $b = 2a + 1$.
- (5) Which of the following subsets of $\mathbb{R}^2$ are subspaces of $\mathbb{R}^2$? The set W of all vectors of the form
 - (a) The union of the first and the third quadrant.
 - (b) The set of all points in the unit disk (on and inside the unit circle).

- (6) Which of the following subsets of P_2 are subspaces of P_2 ? The set W of all polynomials of the form
- (a) $a_0 + a_1t + a_2t^2$, where $a_0 = a_1 = 0$.
 - (b) $a_0 + a_1t + a_2t^2$, where $a_1 = 2a_0$.
 - (c) $a_0 + a_1t + a_2t^2$, where $a_0 + a_1 + a_2 = 2$.
 - (d) $a + t^2$.
 - (e) $a_0 + a_1t + a_2t^2$, where a_0, a_1, a_2 are integers.
 - (f) $p(t) = a_0 + a_1t + a_2t^2$, where $p(0) = 0$.
- (7) Which of the following subsets of M_{23} are subspaces of M_{23} ? The set W of all matrices of the form
- (a) $\begin{bmatrix} a & b & c \\ d & 0 & 0 \end{bmatrix}$, where $b = a + c$.
 - (b) $\begin{bmatrix} a & b & c \\ d & 0 & 0 \end{bmatrix}$, where $c > 0$.
 - (c) $\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$, where $a = -2c$ and $f = 2e + d$.
- (8) Which of the following subsets of M_{nn} are subspaces of M_{nn} ?
- (a) The set of all $n \times n$ symmetric matrices.
 - (b) The set of all $n \times n$ nonsingular matrices.
 - (c) The set of all $n \times n$ diagonal matrices.
 - (d) The set of all $n \times n$ singular matrices.
 - (e) The set of all $n \times n$ upper triangular matrices.
 - (f) The set of all $n \times n$ matrices whose determinant is 1.
- (9) Which of the following subsets are subspaces of $C(-\infty, \infty)$
- (a) All nonnegative functions.
 - (b) All constant functions.
 - (c) All functions f such that $f(0) = 0$.
 - (d) All functions f such that $f(0) = 5$.
 - (e) All differentiable functions.

(10) Let $S = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$. Determine if $u = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ belongs to $\text{span } S$.

(11) Let $S = \left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \\ 6 \end{bmatrix} \right\}$. Determine if $u = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$ belongs to $\text{span } S$ (i.e. if it's a linear combination of the vectors of S). How many vectors are in $\text{span } S$?

(12) Let W be the set of all vectors of the form $\begin{bmatrix} s \\ 3s \\ 2s \end{bmatrix}$. Show that W is a subspace of $\mathbb{R}^3$ by finding a vector v in $\mathbb{R}^3$ such that $W = \text{span}\{v\}$.

(13) Let W be the set of all vectors of the form $\begin{bmatrix} 5a + 2b \\ a \\ b \end{bmatrix}$. Show that W is a subspace of $\mathbb{R}^3$ by finding vectors u and v in $\mathbb{R}^3$ such that $W = \text{span}\{u, v\}$.

(14) Let W be the set of all vectors of the form shown (a , b , and c are arbitrary real numbers). Find a set of vectors S that span W

(a) $\begin{bmatrix} 3a + b \\ 4 \\ a - 5b \end{bmatrix}$.

(b) $\begin{bmatrix} a - b \\ b - c \\ c - a \\ b \end{bmatrix}$.