

Review

Review the following: definition of functions, domain, target, range, image, preimage, one-to-one (injective), onto (surjective), bijection (one-to-one correspondence), composition of functions.

Review from Calculus and Discrete Math:

Prove or disprove the following: Let $f : A \longrightarrow B$ and $g : B \longrightarrow C$ be functions, let $\phi \neq S \subseteq A$ and $\phi \neq T \subseteq B$, let G be preimage of the image of S and H the image of the preimage of T , and let a, b, c , and d be real numbers and let $a < b$ and $c < d$. (For the first three parts, if the statement is false and if one direction is true, indicate that, and give a counterexample to show the other direction is false.) Then

- (1) $g \circ f$ is one-to-one iff g and f are one-to-one.
- (2) $g \circ f$ is onto iff g and f are onto.
- (3) $g \circ f$ is bijection iff g and f are bijection.
- (4) $G = S$. If false, indicate if one of these two sets is a subset of the other.
- (5) $H = T$. If false, indicate if one of these two sets is a subset of the other.
- (6) Prove that the function $f : (\mathbb{N} \cup \{0\}) \times (\mathbb{N} \cup \{0\}) \longrightarrow \mathbb{N} \cup \{0\}$, defined by $f(k, n) = 2^k(2n + 1) - 1$ is one-to-one.
- (7) Give an example of a bijection from (a, b) to (c, d) .
- (8) Give an example of a bijection from $\mathbb{R}$ to (a, ∞) .
- (9) Give an example of a bijection from $(0, 1)$ to (a, ∞) .

Definitions: Let V and W be vector spaces and let L be a function from V into W . L is called a *linear transformation* if it satisfies the following two conditions:

- (i) $L(u + v) = L(u) + L(v)$, for all u and v in V .
- (i) $L(\beta u) = \beta L(u)$, for every u in V and every scalar β .

Remarks:

- (1) If $W = V$, we call L a linear operator on V .

(2) The above two conditions can be replaced by:

$L(\beta u + \gamma v) = \beta L(u) + \gamma L(v)$, for all u and v in V and for all scalars β and γ .

(3) Some people write Lu instead of $L(u)$.

Definition: Let L be a linear transformation from a vector space V into a vector space W and M be a linear transformation from W into a vector space X .

- (1) If $W = V$ and $L(v) = v$ for all $v \in V$, then L is called the identity operator on V and it's denoted by I .
- (2) range L (the range of L ; i.e. $L(V)$) = $\{w \in W \mid w = L(v), \text{ for some } v \in V\}$.
- (3) If range $L = W$, we say L is onto.
- (4) L is called one-to-one iff for all v_1 and v_2 in V , if $L(v_1) = L(v_2)$, then $v_1 = v_2$.
- (5) ker L (the kernel of L) = $\{v \in V \mid L(v) = 0\}$.
- (6) The dimension of ker L is called the nullity of L , denoted nullity L .
- (7) The dimension of range L is called the rank of L , denoted rank L .
- (8) If $S \subseteq V$, img S (the image of S ; i.e. $L(S)$) = $\{w \in W \mid w = L(v), \text{ for some } v \in S\}$.
- (9) If $T \subseteq W$, preimage T (the preimage of T) is $\{v \in V \mid L(v) \in T\}$.
- (10) The product ML is the function defined as follows: $ML(u) = (M \circ L)(u) = M(L(u))$, $\forall u \in V$.

Theorem: Let L be a linear transformation from a vector space V into a vector space W and let M be a linear transformation from W into a vector space X . Then

- (1) $L(0) = 0$.
- (2) ML is a linear transformation.
- (3) $L(u - v) = L(u) - L(v)$, for all u and v in V .
- (4) If $S = \{v_1, v_2, \dots, v_k\}$ spans V , then $T = L(S) = \{L(v_1), L(v_2), \dots, L(v_k)\}$ spans range L . (In particular, if S is a basis for V and $u \in \text{range } L$, then u is completely determined by members of T).
- (5) The image of a subspace of V is a subspace of W . In particular, range L is a subspace of W .

- (6) The preimage of a subspace of W is a subspace of V .
- (7) If L_2 is also a linear transformation from V into W and $\{v_1, v_2, \dots, v_n\}$ is a basis for V . If $L(v_i) = L_2(v_i)$ for $i = 1, 2, \dots, n$, then $L(u) = L_2(u)$, for each $u \in V$.
- (8) $\ker L$ is a subspace of V .
- (9) $\ker L = \{0\}$ if and only if L is one-to-one if and only if the image of every linearly independent subset of V is linearly independent in W .
- (10) $\text{rank } L + \text{nullity } L = \dim V$.
- (11) If $\dim V = \dim W$. Then L is one-to-one iff L is onto iff the image of every basis for V is a basis for W .
- (12) If L is onto, then $\dim W \leq \dim V$.
- (13) If L is a bijection, then the image of every basis for V is a basis for W .

Theorem: Let L be a function from a vector space V into a vector space W . If $L(0) \neq 0$, then L is not a linear transformation.

Remark: Read the examples in Section 10.1 especially those about projection, rotation, dilation, contraction, and reflection. These are important.

Examples of linear transformations:

- (1) (Projection) $L : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$, $L(x, y, z) = (x, y)$.
- (2) (Dilation) $L : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$, $L(u) = ru$, $r > 1$.
- (3) (Contraction) $L : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$, $L(u) = ru$, $0 < r < 1$.
- (4) (Reflection with respect to the x-axis) $L : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, $L(x, y) = (x, -y)$.
- (5) (Rotation counterclockwise by θ) $L : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, $L(u) = Au$, where

$$\begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}.$$

Exercises of linear transformations: Find the matrix of the linear transformation $L : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ and $L(x, y)$ if

- (1) $L(u)$ is rotation of u clockwise by θ .

- (2) $L(u)$ is reflection of u about the line $y = x$.
- (3) $L(u)$ is reflection of u about the line $y = -x$.
- (4) $L(u)$ is reflection of u about the origin.
- (5) $L(u)$ is reflection of u about the x -axis.
- (6) $L(u)$ is reflection of u about the y -axis.