# Examples

(1) Show that $f_1(x) = x^2$ and $f_2(x) = x|x|$ are lineary independent in $C[-1, 1]$.

Solution: Note first that $|x|$ is not differentiable at $x = 0$, and hence, the Wrnoskian method does not apply to any set of functions defined on $[-1, 1]$ that include $f(x) = |x|$, because $|x| \notin C^1[-1, 1]$. But, it applies to any set of functions in $C[a, b]$, such that $-1 \notin [a, b]$. But, in our case we don't have to worry about that because the function we have is $x|x|$ not $|x|$. Multiplying $|x|$ by $x$ removes the sharp corner that $|x|$ has at $x = 0$, because $x|x| = x^2$ if $x \geq 0$ and $-x^2$ if $x < 0$. Thus, the derivative of $x|x|$ is $2x$ if $x \geq 0$ and $-2x$ if $x < 0$, or equivalently, the derivative of $x|x|$ is $2|x|$. Now since $f_1(x)$ and $f_2(x)$ are in $C^1[-1, 1]$, we can apply the Wronskian method to check their linear independence. When we do that we get the determinant is zero. But that does not mean the functions are linearly independent; it means we cannot reach any conclusion based on this theorem. Hence, we have to try another method. This is the method we did in class (see your notes).

(2) Which of the following sets span $\mathbb{R}^3$? If the given set spans $\mathbb{R}^3$, write $(1, 1, 1)$ as a linear combination of the vectors in the set. If the set does not span $\mathbb{R}^3$, give an example of a vector in $\mathbb{R}^3$ that does not belong to the span of the set:

(a) $S = \{v_1 = (1, 0, 0), v_2 = (0, 0, 1), v_3 = (2, 1, 0), v_4 = (0, 1, 3)\}$.

Solution: If the given set spans $\mathbb{R}^3$, then every vector $v = (a, b, c)$ in $\mathbb{R}^3$ can be written as a linear combination of the vectors in $S$. I.e., for every $a, b, c \in \mathbb{R}$, we must be able to find constants $c_1, c_2, c_3$ such that $c_1v_1 + c_2v_2 + c_3v_3 = v$. In other words, the system $c_1v_1 + c_2v_2 + c_3v_3 = v$ must have a solution for every vector $v \in \mathbb{R}^3$ (i.e. it must have a solution for all values of $a$, $b$, and $c$). Note that the unkowns in this system are $c_1$, $c_2$, and $c_3$. This means if the system is inconsistent (i.e. has no solution) for some values of $a$, $b$, and $c$, then $S$ does not span $\mathbb{R}^3$. So, let's find out whether the system is consistent (i.e. has at least one solution) for all values of $a$, $b$, and $c$. First form the augmented matrix of the system which is

$$\left[\begin{array}{cccc|c} 1 & 0 & 2 & 0 & a \\ 0 & 0 & 1 & 1 & b \\ 0 & 1 & 0 & 3 & c \end{array}\right].$$

Now find the row echelon form or the reduced row echelon form (we'll find the row echelon form) of the above matrix, which is

$$\left[\begin{array}{cccc|c} 1 & 0 & 2 & 0 & a \\ 0 & 1 & 0 & 3 & c \\ 0 & 0 & 1 & 1 & b \end{array}\right].$$

The resulting system is $c_3 + c_4 = b$, $c_2 + 3c_4 = c$, $c_1 + 2c_3 = a$. Solve to get $c_1 = a - 2c_3$, $c_4 = b - c_3$, $c_2 = c - 3c_4 = c - 3b + 3c_3$, and $c_3$ is arbitrary. Thus, the system is consistent for any vector $v$ (i.e. for all values of $a$, $b$, and $c$). Therefore, $S$ spans $\mathbb{R}^3$.

Thus, we can write $(1, 1, 1)$ as a linear combination of the vectors in $S$. To do that, just pick a value for $c_3$ and substitute 1 for $a$, $b$, and $c$. If we take $c_3 = 0$, we get $c_1 = 1$, $c_2 = -2$, $c_4 = 1$. Thus, $(1, 1, 1) = 1v_1 - 2v_2 + 0v_3 + 1v_4 = v_1 - 2v_2 + v_4$. Note that $S$ is lineary dependent (because it has 4 vectors which means it has one vector more than the dimension of $\mathbb{R}^3$, and since 3 lin. indep. vectors are enough to span $\mathbb{R}^3$, then we can get rid of exactly one vector of $S$ and still get the same span (remember $S$ is linearly dependent implies a vector in $S$ can be written as a linear combination of the remaining vectors in $S$). By taking $c_3 = 0$, we decided to eliminate $v_3$.

(b) $\{v_1 = (1, 2, -1), v_2 = (6, 3, 0), v_3 = (4, -1, 2), v_4 = (2, -5, 4)\}$.

Solution: Follow the procedure we did in the previous part. The augmented matrix is

$$\left[\begin{array}{cccc|c} 1 & 6 & 4 & 2 & a \\ 2 & 3 & -1 & -5 & b \\ -1 & 0 & 2 & 4 & c \end{array}\right].$$

The row echelon form of the above matrix is

$$\begin{bmatrix} 1 & 6 & 4 & 2 & a \\ 0 & 1 & 1 & 1 & (2a-b)/9 \\ 0 & 0 & 0 & 0 & c+a-(2/3)(2a-b) \end{bmatrix}.$$

Note that the above system is inconsistent when $c+a-(2/3)(2a-b) \neq 0$. Thus, any vector $v = (a, b, c)$ such that $c + a - (2/3)(2a - b) \neq 0$ is not in the span of $S$ (which means $S$ does not span $\mathbb{R}^3$). For example, if we take $a = b = 0$ and $c = 1$, then $c + a - (2/3)(2a - b) = 1 \neq 0$. Thus, $(0, 0, 1) \notin \text{span } S$.

(c) $\{(1, 1, 2), (2, 3, 4)\}$.

Solution: No, because we need at least 3 vectors to span $\mathbb{R}^3$ because *dim* $\mathbb{R}^3 = 3$.

(3) Which of the following sets span $P_2$? If the given set spans $P_2$, write $1+t+t^2$ as a linear combination of the vectors in the set. If the set does not span $P_2$, give an example of a vector in $P_2$ that does not belong to the span of the set:

(a) $S = \{v_1 = 1, v_2 = t^2, v_3 = 2 + t, v_4 = t + 3t^2\}$.

Solution:

When you write this as a system, you get the same augmented matrix we obtained in the first part of the previous question. Thus, $S$ spans $P_2$ and $1 + t + t^2 = v_1 - 2v_2 + v_4$.

(b) $\{v_1 = 1 + 2t - t^2, v_2 = 6 + 3t, v_3 = 4 - t + 2t^2, v_4 = 2 - 5t + 4t^2\}$.

Solution: When you write this as a system, you get the same augmented matrix we obtained in the second part of the previous question. Thus, $S$ does not span $P_2$ and $t^2$ is not in the span of $S$.

(c) $\{1 + t + 2t^2, 2 + 3t + 4t^2\}$.

Solution: No, because we need at least 3 vectors to span $P_2$ because *dim* $P_2 = 3$.

(4) Determine if each of the following sets is linearly independent in $\mathbb{R}^4$. If not, write one of the vectors in $S$ as a linear combination of the other vectors in $S$. Also, determine if $S$ is a basis for $\mathbb{R}^4$. If yes, write $(1, 2, 3, 4)$ as a linear combination of the vectors in $S$:

(a) $S = \{v_1 = (1,1,2,1), v_2 = (1,0,0,2), v_3 = (4,6,8,6), v_4 = (0,3,2,1)\}$.

(b) $S = \{v_1 = (4,2,-1,3), v_2 = (6,5,-5,1), v_3 = (2,-1,3,5)\}$.

(c) $S = \{v_1 = (1,1,1,1), v_2 = (2,3,1,2), v_3 = (3,1,2,1), v_4 = (2,2,1,1)\}$.

(5) Determine if each of the following sets is linearly independent in $P_3$. If not, write one of the vectors in $S$ as a linear combination of the other vectors in $S$. Also, determine if $S$ is a basis for $P_3$. If yes, write $1+2t+3t^2+4t^3$ as a linear combination of the vectors in $S$:

(a) $S = \{v_1 = 1+t+2t^2+t^3, v_2 = 1+2t^3, v_3 = 4+6t+8t^2+6t^3, 3t+2t^2+t^3\}$.

(b) $S = \{v_1 = 4+2t-t^2+3t^3, v_2 = 6+5t-5t^2+t^3, v_3 = 2-t+3t^2+5t^3\}$.

(c) $S = \{v_1 = 1+t+t^2+t^3, v_2 = 2+3t+t^2+2t^3, v_3 = 3+t+2t^2+t^3, v_4 = 2+2t+t^2+t^3\}$.