

Norms, Orthogonal and Orthonormal

Definitions: Let v be an $n \times 1$ vector in $\mathbb{C}^n$ and let p be a finite positive real number. The p norm of v , denoted $\|v\|_p$, is defined as follows:

$$\|v\|_p = \left(\sum_{i=1}^n |v_i|^p \right)^{1/p}.$$

The ∞ – norm of p , denoted $\|p\|_\infty$, is defined as follows:

$$\|v\|_\infty = \max_{1 \leq i \leq n} |x_i|.$$

In particular, if $v \in \mathbb{R}^n$, then

$$\|v\|_2 = \left(\sum_{i=1}^n (v_i)^2 \right)^{1/2}.$$

$$\|v\|_\infty = \max_{1 \leq i \leq n} |x_i|.$$

$$\|v\|_1 = \sum_{i=1}^n |v_i|.$$

Notes:

- (1) If we write just $\|u\|$ without specifying the norm, then we mean any norm. In the future, we'll be using only the two-norm. Thus, when we write $\|u\|$ in the future, we mean $\|u\|_2$.
- (2) The two-norm is also called the Euclidean norm.
- (3) Think of norm as length or magnitude.

Remarks, facts, and definitions: Let u and v be in $\mathbb{R}^n$, let $\alpha \in \mathbb{R}$, and let θ be the angle between u and v . Then

- (1) $\|u\| \geq 0$ and $\|u\| = 0$ iff $u = 0$.
- (2) $\|u + v\| \leq \|u\| + \|v\|$ (triangle inequality).
- (3) $\|\alpha u\| = |\alpha| \|u\|$.
- (4) $u \cdot u \geq 0$ and $u \cdot u = 0$ iff $u = 0$.
- (5) $u \cdot v = v \cdot u$.

- (6) $u \cdot v = \|u\|_2 \|v\|_2 \cos(\theta)$.
- (7) $|u \cdot v| \leq \|u\|_2 + \|v\|_2$.
- (8) $\|u\|_2 = \sqrt{u \cdot u} = \sqrt{u^T u}$.
- (9) u and v are orthogonal iff $u \cdot v = 0$, parallel iff $|u \cdot v| = \|u\| \|v\|$.
- (10) Note that u and v are orthogonal iff $\cos(\theta) = 0$, parallel iff $\cos(\theta) = \pm 1$, and in the same direction iff $\cos(\theta) = 1$.
- (11) u is called a unit vector (or a vector of norm/length 1) iff $\|u\| = 1$.
- (12) $\frac{u}{\|u\|}$ is a unit vector in the same direction as u .
- (13) u and v are called orthonormal iff $\|u\| = \|v\| = 1$ (i.e. they are unit vectors) and they are orthogonal.

Orthogonal and Orthonormal Sets and Bases

Definitions and Facts

- (1) A set $S = \{u_1, u_2, \dots, u_k\}$ is called orthogonal iff any two distinct vectors in S are orthogonal. I.e. $u_i \cdot u_j = 0$ for $i \neq j$.
- (2) A subset $S = \{u_1, u_2, \dots, u_k\}$ is called orthonormal iff it's orthogonal and every vector in S is a unit vector. I.e. $u_i \cdot u_j = 0$ for $i \neq j$ and $u_i \cdot u_i = 1$, $i = 1, \dots, k$.
- (3) An orthogonal set is linearly independent.
- (4) An orthonormal set is linearly independent.
- (5) A set S is called an orthogonal basis for a vector space V iff S is orthogonal and S is a basis for V .
- (6) A set S is called an orthonormal basis for a vector space V iff S is orthonormal and S is a basis for V .
- (7) Let $S = \{u_1, u_2, \dots, u_k\}$ be an orthogonal basis for a vector space V and let v be any vector in V . Then $v = c_1 u_1 + c_2 u_2 + \dots + c_k u_k$, where $c_j = \frac{v \cdot u_j}{u_j \cdot u_j}$, $1 \leq j \leq k$.
- (8) Let $S = \{u_1, u_2, \dots, u_k\}$ be an orthonormal basis for a vector space V and let v be any vector in V . Then $v = c_1 u_1 + c_2 u_2 + \dots + c_k u_k$, where $c_j = v \cdot u_j$, $1 \leq j \leq k$.