Row Space, Column Space, and Rank

Let A be an $m \times n$ real matrix

(1) The row space of A is the subspace of $\mathbb{R}^n$ spanned by the rows of A. The dimension of the row space of A is called the row rank of A.

(2) The column space of A is the subspace of $\mathbb{R}^m$ spanned by the columns of A. The dimension of the column space of A is called the column rank of A.

(3) To find a basis for the row space of A and its dimension, you can use one of the following two methods:

   (a) Find the reduced row echelon form of A (call it B). Then the nonzero rows of B (taken as row vectors) form a basis for the row space of A, and the dimension of the row space of A is the number of these nonzero rows. This procedure will give you a basis that is not necessarily formed from rows from A, but such a basis has nice properties similar to those of the standard basis. *Note that the columns of A corresponding to the leading 1's in B form a basis for the column space of A and this basis contains only columns from A. But such a basis does not have nice properties similar to those of the standard basis.*

   (b) Write the rows of A as columns to form the matrix A^T. Then find the reduced row echelon form of A^T (call it B). Then, take the columns (write them as row vectors) of A^T corresponding to the leading 1's in B. The dimension of the row space of A is the number of these leading 1's. This procedure will give you a basis that is formed from rows from A, but such a basis does not have nice properties similar to those of the standard basis. *Note that the rows of B (taken as column vectors) form a basis for the column space of A and this basis does not necessarily contain columns of A, but it has nice properties like the standard basis.*

(4) To find a basis for the column space of A and its dimension, you can use one of the following two methods:

   (a) Write the columns of A as rows to form the matrix A^T. Then find the reduced row echelon form of A^T (call it B). Then the nonzero rows of B

(written as column vectors) form a basis for the column space of A, and the dimension of the column space of A is the number of these nonzero rows. This procedure will give you a basis that is not necessarily formed from columns from A, but such a basis has nice properties similar to those of the standard basis. *Note that the columns of A^T (written as row vectors) corresponding to the leading 1's in B form a basis for the rows space of A and this basis contains only rows from A.*

(b) Find the reduced row echelon form of A (call it B). Then, take the columns (write them as column vectors) of A corresponding to the leading 1's in B. The dimension of the column space of A is the number of these leading 1's. This procedure will give you a basis that is formed from columns from A, but such a basis does not have nice properties similar to those of the standard basis. *Note that the nonzero rows of B (taken as row vectors) form a basis for the row space of A and this basis does not necessarily contain rows from A, but it has nice properties like the standard basis.*

(5) If you're given a matrix A and you're asked to find two bases for the column space of A and two bases for the row space of A, then do the following:

(a) Find the reduced row echelon form of A. That will give you a basis for the row space of A that may not contain rows from A, and a basis for the column space of A that contains only columns from A.

(b) Find the reduced row echelon form of A^T. That will give you a basis for the row space of A that consists of rows from A, and a basis for the column space of A that may not contain columns from A.

(6) The row rank of A is equal to the column rank of A, and this is denoted by rank A or rank(A).

(7) To find the rank of a matrix A, find the reduced row echelon form of A (call it B). Then, rank A is equal to the number of nonzero rows of B.

(8) If A is row equivalent to B, then the rwo spaces of A and B are equal, and hence, rank $A =$ rank B.

(9) If A is an $m \times n$ matrix, then rank $A +$ nullity $A =$ n.

(10) Note that columns in A^T are rows in A and rows in A^T are columns in A. Thus, the row space of A^T is the same as the column space of A and the column space of A^T is the same as the row space of A. But, remember you need to change row vectors taken from A^T to column vectors in A and column vectors taken from A^T to row vectors in A.

(11) Let A be an $n \times n$ matrix. Then the following are equivalent:
   (a) A is nonsingular.
   (b) $\det(A) \neq 0$.
   (c) The system $Ax = 0$ has only the trivial solution.
   (d) The system $Ax = b$ has a unique solution for any $n \times 1$ vector.
   (e) A is row equivalent to I.
   (f) nullity $A = 0$.
   (g) rank $A = n$.
   (h) The columns of A are linearly independent.
   (i) The rows of A are linearly independent.