Linear Systems of Equations, Inverses, Row Equivalence,

Row Echelon Form, Reduced Row Echelon Form

Definitions: Let $A = (a_{ij})$ be an $n \times n$ matrix:

(1) The *minor* of element a_{ij} is the determinant of the matrix obtained from A by deleting row i and column j.

(2) The *cofactor* of element a_{ij}, denoted A_{ij}, is $(-1)^{i+j}$ multiplied by the minor of a_{ij}.

(3) The *adjoint* of A, denoted $\text{adj}(A)$ is $\text{adj}(A) = (A_{ji}) = (A_{ij})^T$.

Let A and B be $n \times n$ matrices:

(1) A linear system can have either one solution or no solutions or infinitely many solutions. If the system has a solution, we say it's *consistent*. If the system has no solution, we say it's *inconsistent*.

(2) A homogeneous linear system of equations can have either a unique solution which is the trivial solution (the zero vector) or infinitely many solutions. Thus, it's always consistent.

(3) A linear system of equations can be solved by using elimination or substitution or Gaussian elimination or Gauss-Jordan elimination. If the coefficient matrix is square and nonsingular, then the solution can be obtained also by Gramer's rule or by multiplying the inverse of the coefficient matrix with the vector of constant terms.

(4) An upper-triangular system can be solve by backward substitution and a lower-triangular system can be solved by forward substitution.

(5) To solve a linear system by Gaussian elimination, form the augmented matrix (by taking the coefficient matrix and following it with the vector of constant terms) and transform it to row echelon form. Then solve the new system with backward substitution. If the system is homogeneous, there is no need to include the vector of constant terms which is the zero vector because the zero vector will remain zero when elementary row operations are performed.

(6) To solve a linear system by Gauss-Jordan elimination, form the augmented matrix (by taking the coefficient matrix and following it with the vector of constant terms) and transform it to reduced row echelon form. Then solve the new system.

(7) To solve a linear system for more than one vector of constant terms by using Gaussian elimination or Gauss-Jordan elimination, put all vectors of constant terms in the augmented matrix after the coefficient matrix.

(8) To get the determinant of a matrix, several methods can be used including cofactors, Gaussian elimination, and Gauss-Jordan elimination. When the latest two methods are used, the following must be kept in mind:
   (a) Interchanging two rows multiplies the determinant by -1.
   (b) Multiplying a row by a nonzero constant multiplies the determinant by that number.
   (c) Adding a multiple of a row to another of the forma $r_k + \alpha r_m \longrightarrow r_k$ has no effect on the determinant. But, $r_m + \alpha r_k \longrightarrow r_k$ does change the determinant (it multiplies the determinant by α).

   To use cofactors to find the determinant, the determinant can be expanded along any row or any column; i.e.

$$\det(A) = \sum_{j=1}^{n} a_{ij} A_{ij}, \ i = 1, \ \cdots, \ n.$$

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(9) To find the inverse of an $n \times n$ nonsingular matrix, several methods can be used. One of those is by using the adjoint and another by using Gauss-Jordan elimination. Using the adjoint, $A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A)$. To use Gauss-Jordan, form the matrix $M = [A|I_n]$ and perform elementary row operations on M to get the reduced row echelon form of M. Call the reduced row echelon form of M N. Then, $N = [C|D]$, where C and D are $n \times n$ each. If $C = I_n$, then $A^{-1} = D$. If $C \neq I_n$, then A is singular. In this case, C must have at least one zero row.

(10) Note that finding the inverse by Gauss-Jordan as described above is equivalent to solving n linear systems of equations. The vectors of constant terms in this case are e_i, $i = 1, \cdots, n$, where e_i is the vector whose i^{th} entry is 1 and the remaining entries are zeros. So, actually the second part of the augmented matrix, which is I_n, consists of the vectors of constant terms. So, what we do actually is that we solve the n systems $Ax = e_i$, $i = 1, \cdots, n$, by Gauss-Jordan elimination. The solution of $Ax = e_1$ gives us the first column of A^{-1}, the solution of $Ax = e_2$ gives us the second column of A^{-1}, and so on.

(11) Row equivalence is an equivalence relation on matrices.

(12) If two matrices are row equivalent, they don't necessarily have the same determinant.

(13) If AB is defined, BA is not necessarily defined. Even if BA is also defined, it is not necessarily equal to AB and it may even have a different size than AB.

(14) Let A and B be matrices such that the following operations are defined and let p and q be integers, and α a complex number. Then
   (a) If p is positive, then $A^p = AA^{p-1} = A^{p-1}A$, and if A is nonsingular, then $A^{-p} = (A^p)^{-1}$.
   (b) $(A^p)^T = (A^T)^p$.
   (c) $(AB)^T = B^T A^T$.
   (d) $(A^T)^T = A$.
   (e) $(A + B)^T = A^T + B^T$.
   (f) $(\alpha A)^T = \alpha A^T$.
   (g) If A is nonsingular, then $(A^{-1})^{-1} = A$, $(A^T)^{-1} = (A^{-1})^T$, and $(A^p)^{-1} = (A^p)^{-1}$.
   (h) If A is nonsingular and $\alpha \neq 0$, then αA is nonsingular and $(\alpha A)^{-1} = \frac{1}{\alpha} A^{-1}$. In particular, $(-A)^{-1} = -(A^{-1})$.
   (i) If A and B are nonsingular, then so is AB and $(AB)^{-1} = B^{-1}A^{-1}$.
   (j) Assume A, B, and $A + B$ are invertible. In general, $(A + B)^{-1} \neq A^{-1} + B^{-1}$.

(15) In general matrices and vectors do not satisfy the zero factor property. I.e. you can find vectors a and b such that their dot/inner product is zero but none of them is the zero vector, and you can find two matrices A and B such that AB is the zero matrix but none of them is the zero matrix. Moreover, if $A^2 = A$, that does not necessarily imply $A = I$ or $A = 0$. But if A is nonsingular, then $A^2 = A$ iff $A = I$.

(16) A can be written as a sum of a symmetric matrix which is $\frac{1}{2}(A + A^T)$ and a skew-symmetric matrix which is $\frac{1}{2}(A - A^T)$.

(17) The following are equivalent:

(a) $\det(A) \neq 0$.

(b) A is nonsingular.

(c) A is row equivalent to I_n.

(d) $Ax = b$ has a unique solution for every $n \times 1$ vector b.

(e) $Ax = 0$ has only the trivial solution.