Functions

Definition 1. A *function* f is a binary relation from a set A to a set B such that for every $a \in A$, there is exactly one $b \in B$ such that $(a, b) \in f$. A is called the *domain* of f and B is called the *target* of f. We write $f: A \longrightarrow B$.

Remarks: Let $f: A \longrightarrow B$ be a function. Then

(1) If $(a, b) \in f$, then we say $f(a) = b$. In this case, b is said to be an *image* of a and a is said to be a *preimage* of b.

(2) Let $X \subseteq A$ and $Y \subseteq B$. Then
   (a) The *image* of X, denoted $f(X)$, is the set of images of the elements of X. I.e. $f(X) = \{b \in B \mid b = f(a) \text{ for some } a \in X\}$. Some people define that as $f(X) = \{f(x) \mid x \in X\}$. People who use this definition assume repetition is removed from $f(X)$ (note that it's possible for two elements (or more) in X to share the same image).
   (b) The preimage of Y, denoted $f^{-1}(Y)$ (this does not mean the inverse of f exists; it's just a notation), is $f^{-1}(Y) = \{a \in A \mid f(a) \in Y\}$.

(3) The *range/image* of f, denoted *rng* f or *rng*(f) is $f(A)$. I.e. *rng* $f = \{b \in B \mid b = f(a) \text{ for some } a \in A\}$. The domain of f is denoted *dom* f or *dom*(f).

(4) A function is also called a *mapping* or *transformation*.

(5) Every function is a binary relation, but not every binary relation is a function.

The definition of a function is usually stated as follows: $f: A \longrightarrow B$ is a function if every element in A has an image in B and whenever $x_1 = x_2$, we must have $f(x_1) = f(x_2)$.

Example 2. Let $A = \{a, b, c\}$, let $B = \{1, 2, 3\}$, and let $R_1 = \{(a, 1), (b, 2), (c, 2)\}$, $R_2 = \{(a, 1), (b, 2)\}$, and $R_3 = \{(a, 1), (a, 2), (b, 2), (c, 3)\}$. Then, R_1 defines a function, but R_2 and R_3 do not. R_2 is not a function, because c has no image in B, and R_3 is not a function, because a has two different images.

Definition 3. Let X be a set. The *identity* function on X, denoted 1_X, is defined to be $1_X(x) = x$, for all $x \in x$. I.e. 1_X maps every element of the domain to itself.

Definition 4. A function $f: A \longrightarrow B$ is said to be *one-to-one*, denoted 1-1, (or *injective*) if whenever $f(x_1) = f(x_2)$, we must have $x_1 = x_2$.

In other words, f is one-to-one if no two points in the domain share the same image.

Remarks:

- To show $f: A \longrightarrow B$ is one-to-one, assume $f(x_1) = f(x_2)$ and prove $x_1 = x_2$.
- To show $f: A \longrightarrow B$ is not one-to-one, all you have to do is to come up with two distinct elements a_1 and a_2 from A such that $f(a_1) = f(a_2)$.

Example 5. Let $A = \{a, b, c\}$, let $B = \{1, 2, 3\}$, and let $R_1 = \{(a, 1), (b, 2), (c, 2)\}$ and $R_2 = \{(a, 3), (b, 1), (c, 2)\}$. Then, R_1 is not 1-1 because $f(b) = f(c)$, but R_2 is 1-1, because no two elements of A have the same image.

Example 6. Let $f: \mathbb{R} \longrightarrow \mathbb{R}$, be $f(x) = x^2$. Then f is not 1-1, because, for example, $f(1) = f(-1)$.

Example 7. Let $f: \mathbb{R}^+ \longrightarrow \mathbb{R}$, be $f(x) = x^2$. Then f is 1-1, because if $f(x_1) = f(x_2)$, for some x_1, x_2 in $\mathbb{R}^+$, then $x_1^2 = x_2^2$, which implies $x_1 = x_2$ or $x_1 = -x_2$. But, since x_1 and x_2 are both positive, the second case cannot happen. Hence, $x_1 = x_2$.

Example 8. Let $f: \mathbb{R}^+ \longrightarrow \mathbb{R}^+$, be $f(x) = x^2$. Then f is 1-1. The explanation is the same as the previous example.

Example 9. Let $f: \mathbb{Z} \longrightarrow \mathbb{Z}$, be $f(x) = 2x - 1$. Then f is 1-1, because if $f(x_1) = f(x_2)$, for some x_1, x_2 in $\mathbb{Z}$, then $2x_1 - 1 = 2x_2 - 1$, which implies $x_1 = x_2$.

Example 10. Let $f: \mathbb{Z} \longrightarrow 2\mathbb{Z}-1$, be $f(x) = 2x-1$. Then f is 1-1. The explanation is the same as the previous example.

Example 11. Prove that the function $f: (\mathbb{N} \cup \{0\}) \times (\mathbb{N} \cup \{0\}) \longrightarrow \mathbb{N} \cup \{0\}$, defined by $f(k, n) = 2^k(2n + 1) - 1$ is one-to-one.

Solution: Assume $f(k_1, n_1) = f(k_2, n_2)$ for some (k_1, n_1) and (k_2, n_2) in $(\mathbb{N} \cup \{0\}) \times (\mathbb{N} \cup \{0\})$. Our goal is to show that $(k_1, n_1) = (k_2, n_2)$. Notice that $(k_1, n_1) = (k_2, n_2)$ iff $k_1 = k_2$ and $n_1 = n_2$. But, since $f(k_1, n_1) = f(k_2, n_2)$, we have that $2^{k_1}(2n_1 + 1) - 1 = 2^{k_2}(2n_2 + 1) - 1$. Now without loss of generality, assume that $k_1 \geq k_2$, and then rewrite the last equation as: $2^{k_1-k_2} = \frac{2n_2+1}{2n_1+1}$. Now notice that the right hand side is either a noninteger fraction or it is odd. On the other hand, the left hand side is either 1 or a multiple of 2. (Notice that $k_1 \geq k_2$ by assumption.) The only way this can happen is if both sides are equal to 1. This yields to the desired result.

DEFINITION 12. A function $f: A \longrightarrow B$ is said to be *onto* (or *surjective*) if for every $b \in B$, there exists $a \in A$, such that $f(a) = b$.

In other words, f is onto if every element in B has a preimage from A, or equivalently, if the target of f is equal to the range of f.

- To show $f: A \longrightarrow B$ is onto, assume $y \in B$ and find $x \in A$ such that $f(x) = y$.
- To show $f: A \longrightarrow B$ is not onto, it suffices to come with a $b \in B$ that has no preimage in A (i.e. there is no $a \in A$ such that $f(a) = b$).

EXAMPLE 13. Let $A = \{a, b, c\}$, let $B = \{1, 2\}$, and let $R_1 = \{(a, 1), (b, 2), (c, 2)\}$ and $R_2 = \{(a, 1), (b, 1), (c, 1)\}$. Then, R_1 is onto, because the range of f is equal to the target of f, but R_2 is not onto. R_2 is not onto, because there is no element in A whose image is 2.

EXAMPLE 14. Let $f: \mathbb{R} \longrightarrow \mathbb{R}$, be $f(x) = x^2$. Then f is not onto, because, for example, -1 is in the target of f, but there is no a in the domain such that $f(a) = -1$. Note that if such an a exists, then we must have $a^2 = -1$. But, this equation has no solution in the domain.

EXAMPLE 15. Let $f: \mathbb{R}^+ \longrightarrow \mathbb{R}$, be $f(x) = x^2$. Then f is not onto for the same reason as the previous example.

EXAMPLE 16. Let $f: \mathbb{R}^+ \longrightarrow \mathbb{R}^+$, be $f(x) = x^2$. Then f is onto, because if b is in the target, then $\sqrt{b}$ is in the domain and $f(\sqrt{b}) = b$.

EXAMPLE 17. Let $f: \mathbb{Z} \longrightarrow \mathbb{Z}$, be $f(x) = 2x - 1$. Then f is not onto, because, for example, 2 is in the range, but there is no a in the domain such that $f(a) = 2$. Note that if such an a exists, then we must have $2a - 1 = 2$, which implies $a = 3/2$, but, 3/2 is not in the domain.

EXAMPLE 18. Let $f: \mathbb{Z} \longrightarrow 2\mathbb{Z}-1$, be $f(x) = 2x-1$. Then f is onto, because if b is in the target, then $\frac{b+1}{2}$ is in the domain (since b is odd, then $b+1$ is even, hence $\frac{b+1}{2}$ is an integer and, hence, in the domain) and $f(\frac{b+1}{2}) = b$.

DEFINITION 19. A function $f: A \longrightarrow B$ is said to be *bijective* or *one-to-one correspondence* if it is one-to-one and onto.

DEFINITION 20. If $f: A \longrightarrow B$ is bijective, then A and B are said to have the same *cardinality*. If this is the case, then we write $|A| = |B|$.

DEFINITION 21. Let f be a function from A to B. If f is a bijection, then the inverse of f, denoted f^{-1}, is defined to be:

$$f^{-1} = \{(y,x) \mid (x,y) \in f\}.$$

Note that $f^{-1}(y) = x$ iff $f(x) = y$. Note also $f^{-1} \neq \frac{1}{f}$.

Remarks: Let $f: A \longrightarrow B$ be a function, and let $X \subseteq A$ and $Y \subseteq B$, and assume f^{-1} exists. Then

(1) f is called *invertible*.
(2) $(f^{-1})^{-1} = f$.
(3) $f(f^{-1}(Y)) \subseteq Y$. **Q:** Suppose f is onto, what can you say?
(4) $X \subseteq f^{-1}(f(X))$. **Q:** Suppose f is one-to-one, what can you say?

Notice that if f is a function from A to B and if f^{-1} is defined, then f^{-1} is a function from B to A.

EXAMPLE 22. Consider $f: \mathbb{R} \longrightarrow \mathbb{R}^+$, defined by $f(x) = e^x$, $\forall x \in \mathbb{R}$. Then, $f^{-1}(x) = \ln(x)$. Notice that the domain of f^{-1} is $\mathbb{R}^+$ and the range is $\mathbb{R}$.

EXAMPLE 23. Consider $f: \mathbb{R} \longrightarrow \mathbb{R}^+$, defined by $f(x) = 10^x$, $\forall x \in \mathbb{R}$. Then, $f^{-1}(x) = \log(x)$. Notice that the domain of f^{-1} is $\mathbb{R}^+$ and the range is $\mathbb{R}$.

EXAMPLE 24. Let $f: \mathbb{R} - \{\frac{1}{2}\} \longrightarrow \mathbb{R} - \{\frac{3}{2}\}$, be defined by:

$$f(x) = \frac{3x}{2x-1}, \ \forall x \in \mathbb{R} - \{\tfrac{1}{2}\}.$$

Then, f^{-1} is a function from $\mathbb{R} - \{\frac{3}{2}\}$ onto $\mathbb{R} - \{\frac{1}{2}\}$, and it's given by:

$$f^{-1}(x) = \tfrac{x}{2x-3},\ \forall x \in \mathbb{R} - \{\tfrac{3}{2}\}.$$

Fact: If $f:\ A \longrightarrow B$ is bijective, then $f^{-1}:\ B \longrightarrow A$ is defined and it is also bijective.

DEFINITION 25. A set A is said to be *countably infinite* if there is a bijection (a bijective function) from A to $\mathbb{N}$ or from $\mathbb{N}$ to A. In other words, A is countable infinite if $|A| = |N|$. A set A is said to be *countable* if it is either *finite* or *countably infinite.*

EXAMPLE 26. Let $A = \{a, b, c\}$, $B = \{1, 2, 3\}$, and define the following functions from A to B.

- $f(a) = 1$, $f(b) = 1$, $f(c) = 2$. Then, f is not one-to-one, because a and b share the same image. Notice that $f(a) = f(b)$. f is not onto, because there is no element in A whose image is 3.
- $f(a) = 1$, $f(b) = 2$, $f(c) = 3$. Then, f is one-to-one and onto.

EXAMPLE 27. The function $f:\ \mathbb{R} \longrightarrow \mathbb{R}$, defined by $f(x) = x^2 - 5x + 6$, is not one-to-one and not onto. Not one-to-one, because, for example, $f(1) = f(4)$. One way to find such points is the following: assume $f(x_1) = f(x_2)$, and then solve the resulting equation, to get: $x_1^2 - 5x_1 + 6 = x_2^2 - 5x_2 + 6$. This implies that $x_1^2 - x_2^2 = 5(x_1 - x_2)$. Thus, $(x_1 - x_2)(x_1 + x_2) = 5(x_1 - x_2)$. Now we have two possibilities: either $x_1 - x_2 = 0$, or $x_1 - x_2 \neq 0$. If $x_1 - x_2 \neq 0$, then we can devide both sides by this quantity, to get $(x_1 + x_2) = 5$.

Now f is not onto, because, for example, there is no $a \in \mathbb{R}$ such that $f(a) = -1$. To prove it, assume that such a exists, then $a^2 - 5a + 6 = -1$. Thus, $a^2 - 5a + 7 = 0$. Try to solve this equation, you get two complex solutions. Thus, since it has no real solutions, then there is no $a \in R$ such that $f(a) = -1$.

EXAMPLE 28. The function $f:\ \mathbb{R} \longrightarrow [\frac{-1}{4}, \infty)$, defined by $f(x) = x^2 - 5x + 6$, is onto but not one-to-one. Not one-to-one, because, for example, $f(1) = f(4)$ (the same reason as the previous example).

Notice that the lowest point on the graph of $f(x)$ is $(\frac{5}{2}, \frac{-1}{4})$. In other words, the graph of f is entirely on or above the line $y = \frac{-1}{4}$. Thus, any horizontal line $y = c$, where $c \geq \frac{-1}{4}$, intersects the graph of f at least once and so f has to be onto. It's easy to prove *"formally"* that f is onto, because if $b \in B$, where $B = [\frac{-1}{4}, \infty)$, then the equation $a^2 - 5a + 6 = b$ has a solution from the domain, $\mathbb{R}$. Notice here that you get $a = \frac{5 \pm \sqrt{1+4b}}{2}$. Now note that since $b \in B$, then $b \geq \frac{-1}{4}$. Thus, $1 + 4b \geq 0$. Therefore, the above equation always has a real solution, say that a_1 is one of them. Then, it follows that $f(a_1) = b$.

EXAMPLE 29. The function $f: (-\infty, \frac{5}{2}] \longrightarrow [\frac{-1}{4}, \infty)$, defined by $f(x) = x^2 - 5x + 6$, is onto and one-to-one. $f(x)$ is onto for the same explanation offered in the previous example. Now f is one-to-one, because if $f(x_1) = f(x_2)$, where x_1 and x_2 are in $A = (-\infty, \frac{5}{2}]$, then $x_1^2 - 5x_1 + 6 = x_2^2 - 5x_2 + 6$, which implies that $(x_1 - x_2)^2 = 5(x_1 - x_2)$. Now we have two cases:

- **Case 1:** $x_1 - x_2 \neq 0$. In this case, we get $x_1 + x_2 = 5$. Notice now that either $x_1 = x_2 = \frac{5}{2}$ or one of them (I mean x_1, x_2) has to be greater than $\frac{5}{2}$, which makes it not in the domain A. So, no two different elements of A can share the same value of f in this case.
- **Case 2:** $x_1 - x_2 = 0$. In this case, we get $x_1 = x_2$.

So, in all cases, no two different elements of A can share the same value of f.

Now notice that since f is one-to-one and onto, then it is bijective, and hence, $(-\infty, \frac{5}{2}]$ and $[\frac{-1}{4}, \infty)$ have the same cardinality.

EXAMPLE 30. The function $f: \mathbb{Z} \longrightarrow \mathbb{Z}$, defined by $f(x) = x^2 - 5x + 6$, is not one-to-one and not onto. Not one-to-one because of the same reasons stated earlier in one of the examples related to this function. Rememeber that you need to set $f(x_1) = f(x_2)$ and then solve the resulting equation. Do so, you'll get $(x_1 - x_2)^2 = 5(x_1 - x_2)$. Thus, either $x_1 = x_2$ or $x_1 + x_2 = 5$. But, the equation $x_1 + x_2 = 5$ has solutions from $\mathbb{Z}$. For example, $x_1 = 1$ and $x_2 = 4$ are solutions. This implies $f(1) = f(4)$.

Now f is not onto, because there is no $a \in \mathbb{Z}$ such that $f(a) = 1$. If such a exists, then we must have $a^2 - 5a + 6 = 1$, which implies $a = \frac{5 \pm \sqrt{5}}{2}$. The two values of a which we get are not integers, because $\sqrt{5}$ is irrational.

EXAMPLE 31. The function $f: \mathbb{Z} \longrightarrow \mathbb{N} \cup \{0\}$, defined by $f(x) = x^2 - 5x + 6$, is not one-to-one and not onto. Not one-to-one and not onto because of the same reasons stated in the previous example.

EXAMPLE 32. We know that $f(x) = e^x$ is a bijective function from $\mathbb{R}$ onto $(0, \infty)$. Try to prove that. This means that $\mathbb{R}$ and $(0, \infty)$ have the same cardinality. Now the questions is: how to get a bijective function from $\mathbb{R}$ onto (b, ∞)? The anwer is by taking $f(x) = b + e^x$. The next questions is: how to get a bijective function from $\mathbb{R}$ onto $(-\infty, b)$? The anwer is by taking $f(x) = b - e^x$.

EXAMPLE 33. Let $f: \mathbb{N} \longrightarrow \mathbb{Z}$ be defined by $f(x) = \frac{x}{2}$ if x is even, and $f(x) = \frac{1-x}{2}$ if x is odd. Then f is bijective (Prove it). Hence, $\mathbb{Z}$ and $\mathbb{N}$ have the same cardinality. Therefore, $\mathbb{Z}$ is countably infinite. Now notice that f^{-1} is defined and it is a bijective function from $\mathbb{Z}$ onto $\mathbb{N}$. Try to find f^{-1}. Does this remind you of anything realted to the homework?

Notation: We will use the set $2\mathbb{N} - 1$ to denote the set of *odd* natural numbers (i.e. the set $\{2n - 1 \mid n \in \mathbb{N}\}$. And we will use the set $2\mathbb{N}$ to denote the set of *even* natural numbers (i.e. the set $\{2n \mid n \in \mathbb{N}\}$.

EXAMPLE 34. The function $f: \mathbb{N} \longrightarrow 2\mathbb{N} - 1$, defined by $f(n) = 2n - 1$ is bijective and so $2\mathbb{N} - 1$ is countably infinite. Thus, $|2\mathbb{N} - 1| = |\mathbb{N}|$.

EXAMPLE 35. The function $f: \mathbb{N} \longrightarrow 2\mathbb{N}$, defined by $f(n) = 2n$ is bijective and so $2\mathbb{N}$ is countably infinite. Thus, $|2\mathbb{N}| = |\mathbb{N}|$.

EXAMPLE 36. The function $f: \mathbb{N} - \{1, 2, 3, 4\} \longrightarrow \mathbb{N}$, defined by $f(n) = n - 4$ is bijective and so $\mathbb{N} - \{1, 2, 3, 4\}$ is countably infinite. Thus, $|\mathbb{N} - \{1, 2, 3, 4\}| = |\mathbb{N}|$.

EXAMPLE 37. The function $f: \mathbb{N} - \{1, 2, 3, 4\} \longrightarrow 2\mathbb{N} - 1$, defined by $f(n) = 2(n - 4) - 1 = 2n - 9$ is bijective and so $|\mathbb{N} - \{1, 2, 3, 4\}| = |2\mathbb{N} - 1| = |\mathbb{N}|$.

EXAMPLE 38. The function $f: \mathbb{N} - \{1, 2, 3, 4\} \longrightarrow 2\mathbb{N}$, defined by $f(n) = 2(n - 4) = 2n - 8$ is bijective and so $|\mathbb{N} - \{1, 2, 3, 4\}| = |2\mathbb{N}| = |\mathbb{N}|$.

EXAMPLE 39. In this example, we will find a bijective function from the interval (a, b) onto the interval (c, d), where none of a, b, c, and d is equal to $\pm\infty$. The function is given by:

$$f(x) = c + \tfrac{d-c}{b-a}(x - a),\ \forall x \in (a, b).$$

Notice that f is not the straight line $y = c + \frac{d-c}{b-a}(x - a)$. f is the segment of the previously- mentioned straight line which lies between $x = a$ and $x = b$.

EXAMPLE 40. Is it possible to find a bijective function from the interval $(1, 2)$ onto the interval $(3, 7)$? If yes, give an example.

Solution:

Yes. The function $f: (1, 2) \longrightarrow (3, 7)$, defined by $f(x) = 4x - 1$, is bijective.

EXAMPLE 41. Is it possible to find a bijective function from the interval $[1, 2)$ onto the interval $[3, 7)$? If yes, give an example.

Solution:

Yes. The function $f: [1, 2) \longrightarrow [3, 7)$, defined by $f(x) = 4x - 1$, is bijective.

EXAMPLE 42. Is it possible to find a bijective function from the interval $[1, 2)$ onto the interval $(3, 7]$? If yes, give an example.

Solution:

Yes. The function $f: [1, 2) \longrightarrow (3, 7]$, defined by $f(x) = 4x - 1$, $\forall x \in (1, 2)$, and $f(1) = 7$, is well-defined and bijective.

Question: Is it possible to find a bijective function from the interval $(1, 2)$ onto the interval $[3, 7]$? If yes, give an example.

Question: Is it possible to find a bijective function from the interval $(1, 2)$ onto the interval $[3, 7)$? If yes, give an example.

Theorem: $|\mathbb{Z}| = |\mathbb{N}|$.

Proof: We have to present a bijection from $\mathbb{Z}$ to $\mathbb{N}$ or from $\mathbb{N}$ to $\mathbb{Z}$, because remember two sets have the same cardinality iff there is a bijection between them. The bijection that we'll present is from $\mathbb{N}$ to $\mathbb{Z}$. The bijection simply maps the set of even natural numbers to the whole set of natural numbers and the set of odd natural numbers to zero and the set of negative integers. In other words, our function is

$$f(x) = \begin{cases} \frac{x}{2} & \text{if } x \text{ is an even natural number} \\ \frac{1-x}{2} & \text{if } x \text{ is an odd natural number} \end{cases}$$

Notice that f is a function from $\mathbb{N}$ to $\mathbb{Z}$. Now we have to prove that f is one-to-one and onto. First, we prove it is one-to-one. So, assume $f(x_1) = f(x_2)$, where x_1 and x_2 are natural numbers. Now we have the following 4 possibilities:

(1) x_1 and x_2 are both even. In this case, $f(x_1) = f(x_2)$ implies $\frac{x_1}{2} = \frac{x_2}{2}$, which implies $x_1 = x_2$.

(2) x_1 and x_2 are both odd. In this case, $f(x_1) = f(x_2)$ implies $\frac{1-x_1}{2} = \frac{1-x_2}{2}$, which implies $x_1 = x_2$.

(3) x_1 is even and x_2 is odd. In this case, $f(x_1) = f(x_2)$ implies $\frac{x_1}{2} = \frac{1-x_2}{2}$, which implies $x_1 + x_2 = 1$. But, x_1 and x_2 are both natural numbers. That means $x_1 \geq 1$ and $x_2 \geq 1$. Thus, $x_1 + x_2 \geq 2$. That means $x_1 + x_2$ cannot be equal to 1. Thus, if x_1 is even and x_2 is odd, then $f(x_1)$ cannot be equal to $f(x_2)$. Remember our assumption that $f(x_1) = f(x_2)$ led to a contradiction (which is $x_1 + x_2 = 1$) . That means the assumption is false. I.e. if x_1 is even and x_2 is odd, then $f(x_1)$ cannot be equal to $f(x_2)$.

(4) x_1 is odd and x_2 is even. This case is similar to the previous case.

Now we'll prove f is onto. To do that, we'll have to prove that if $b \in \mathbb{Z}$, then there exists $a \in \mathbb{N}$ such that $f(a) = b$. Notice that f maps the set of even natural numbers to the set of natural numbers and it maps the set of odd natural numbers to $\mathbb{Z}^- \cup \{0\}$. Now let $b \in \mathbb{Z}$. If $b \in \mathbb{N}$, the preimage of b is $2b$. I.e. $f(2b) = b$. Notice that $2b$ is an even natural number, so we use the first rule (i.e. $f(x) = x/2$). On the other hand, if $b \in \mathbb{Z}^- \cup \{0\}$, the preimage of b is $1 - 2b$. I.e. $f(1 - 2b) = b$. Notice that $1 - 2b$ is an odd natural number, so we use the second rule (i.e. $f(x) = (1 - x)/2$). Now you may wonder how I found those preimages. You'll understand how if you find the inverse (although we haven't proved yet that f is onto, which means we can't conclude at this stage that f is bijection, so we can't talk about the inverse, but it's a good idea to try to find it). Notice first that f^{-1} is a function from $\mathbb{Z}$ to $\mathbb{N}$. Notice also that the first part of f maps the set of even natural numbers to the set of natural numbers. So, the first part of f^{-1} must map the set of natural numbers to the set of even natural numbers. This means we have to work now with the first rule of f (i.e. $f(x) = \frac{x}{2}$). So, assume $b \in \mathbb{N}$, then we want to find an even natural number a such that $f(a) = b$. I.e. we want $\frac{a}{2} = b$. Solve for a (i.e. write a in terms of b) to get $a = 2b$. So, if b is a natural number, then the preimage of b is $2b$ (notice that $2b$ is an even natural number). Now notice that the second part of f maps the set of odd natural numbers to the set of negative integers and zero. So, the second part of f^{-1} must map the set of negative integers and zero to the set of odd natural numbers. This means we have to work now with the second rule of f (i.e. $f(x) = \frac{1-x}{2}$). So, assume $b \in \{0\} \cup \mathbb{Z}^-$, then we want to find an odd natural number a such that $f(a) = b$. I.e. we want $\frac{1-a}{2} = b$. Solve for a (i.e. write a in terms of b) to get $a = 1 - 2b$. So, if b is a negative integer or zero, then the preimage of b is $1 - 2b$ (notice that $1 - 2b$ is an odd natural number).

Example: Try to find f^{-1} using the procedure which we used to prove f is onto. You'll find

$$f^{-1}(x) = \begin{cases} \text{2x} & \text{if } x \text{ is a natural number} \\ \text{1-2x} & \text{if } x \text{ is a negative integer or zero} \end{cases}$$

Notice that f^{-1} is a function from $\mathbb{Z}$ to $\mathbb{N}$. Notice also that f^{-1} is a bijection because f is a bijection.

Facts: $|\mathbb{Q}| = |\mathbb{N}|$, $|\mathbb{R}| = |(0,1)|$, $|\mathbb{R}| \neq |\mathbb{N}|$.

Definition: A set is said to be *countable* if it is either finite or countably infinite.

DEFINITION 43. Let f be a function from X to Y and g a function from Y to Z. Then the composition gof is a function form X to Z, and it is defined as follows: $(gof)(x) = g(f(x))$, for all $x \in X$.

Notice that in order for x to be in the domain of gof, x has to be in the domain of f and $f(x)$ has to be in the domain of g. Thus, it is necessary for the range of f to be contained in the domain of g in order for gof to be defined. If that is not the case, then gof is undefined. Notice also that if $f: A \longrightarrow B$ is invertible; i.e. f^{-1} exists, then $(fof^{-1})(x) = x$, $\forall x \in B$, and $(f^{-1}of)(x) = x$, $\forall x \in A$.

EXAMPLE 44. Let $f: \mathbb{R} \longrightarrow \mathbb{R}$, be $f(x) = (3x+7)^{1/3}$, and let $g: \mathbb{R} \longrightarrow \mathbb{R}$, be $g(x) = 5x-1$. Then, both fog and gof are defined, and $(fog)(x) = (3(5x-1)+7)^{1/3} = (15x+4)^{1/3}$. $(gof)(x) = 5(3x+7)^{1/3} - 1$.

EXAMPLE 45. Show that the cardinality of $\mathbb{R}^+$ is equal to the cardinality of the interval $(0,1)$ by presenting a bijective function from $\mathbb{R}^+$ onto $(0,1)$. Then depend on that function to find a bijective function from $(0,1)$ onto $\mathbb{R}$. Finally, prove that the cardinality of $\mathbb{R}$ is equal to the cardinality of the interval $(0,1)$.

Solution:

Consider the function $f: \mathbb{R}^+ \longrightarrow (0,1)$, defined by $f(x) = \frac{x}{x+1}$, $\forall x \in \mathbb{R}^+$. It is easy to prove that this function is bijective. (The proof is left as an excercise.) Now consider the function $g: \mathbb{R} \longrightarrow \mathbb{R}^+$, defined by $g(x) = e^x$, $\forall x \in \mathbb{R}$. It is easy to prove that this function is bijective. Finally, consider the function $h: \mathbb{R} \longrightarrow (0,1)$, defined by $h(x) = fog(x) = \frac{e^x}{e^x+1}$, $\forall x \in \mathbb{R}$. This function is bijective (why?).

Fact: Let A, B, and C, be sets. If $|A| = |B|$ and $|B| = |C|$, then $|A| = |C|$.

EXAMPLE 46. Show that $|\mathbb{R}| = |(0,1)|$.

Solution: We have shown that $|\mathbb{R}^+| = |(0,1)|$. Now it suffices to show that $|\mathbb{R}| = |\mathbb{R}^+|$, because then we'll have $|\mathbb{R}| = |\mathbb{R}^+| = |(0,1)|$, which implies $|\mathbb{R}| = |(0,1)|$. But, it's easy to see that $f(x) = e^x$ is a bijection between $\mathbb{R}$ and $\mathbb{R}^+$. Hence, $|\mathbb{R}| = |\mathbb{R}^+|$.

Facts: Let $f: A \longrightarrow B$ and $g: B \longrightarrow C$ be functions.

(1) If f and g are invertible, then gof is invertibe and $(gof)^{-1} = f^{-1}og^{-1}$.
(2) $gof = 1_A$ and $gof = 1_B$ iff $g = f^{-1}$ and $f^{-1} = g$ and f and g are bijective.
(3) If gof is one-to-one, then f is one-to-one.

(4) If gof is onto, then g is onto.

(5) If f and g are one-to-one, then gof is one-to-one. The converse is not necessarily true.

(6) If f and g are onto, then gof is onto. The converse is not necessarily true.

(7) If f and g are bijections, then gof is a bijection. The converse is not necessarily true.

EXAMPLE 47. Here is a counterexample that shows the converse of the last 3 items is not true in general: Let $A = \{a, b\}$, $B = \{1, 2, 3\}$, $C = \{\alpha, \beta\}$, and let f and g be defined as follows: $f(a) = 1$, $f(b) = 2$, $g(1) = \alpha$, $g(2) = \beta$, $g(3) = \beta$.

Questions on Functions

- Decide if $f : \mathbb{R}^2 \longrightarrow \mathbb{R} \times \mathbb{R}^+$, $f(x, y) = ((4x + 9)^{1/3}, e^{5y-2})$, is one-to-one and onto.
- Decide if $f : \mathbb{R} \longrightarrow \mathbb{R} \times \mathbb{R}^+$, $f(x) = ((4x + 9)^{1/3}, e^{5x-2})$, is one-to-one and onto.
- Decide if $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$, $f(x, y) = e^{5y-2}$, is one-to-one and onto.
- Define a bijective function from $\mathbb{N} \cup \{-2, -1, 0\}$ onto $2\mathbb{N} - 1$.
- Define a bijective function from $\mathbb{N} \cup \{-2, -1, 0\}$ onto $A - \{1, 3, 5, 7\}$, where $A = 2\mathbb{N} - 1$.
- Let $f(x)$ be defined as follows:

  $$f(x) = \tfrac{4x-1}{3x-7}.$$

  Find the largest possible domain of f and find the range of f corresponding to the domain you've found. Call the domain A. Now find the inverse of f defined (I mean f) over A. What are the domain and range of f^{-1}?
- Let $f(x)$ be defined as follows:

  $$f(x) = \ln(3x - 8).$$

  Find the largest possible domain of f and find the range of f corresponding to the domain you've found. Call the domain A. Now find the inverse of f defined (I mean f) over A. What are the domain and range of f^{-1}?

- Let A be the interval $(4,7)$ and let B be the interval $(2,3)$. Find a bijective function from A onto B.
- Let A be the interval $(4,\infty)$ and let B be the interval $(2,3)$. Find a bijective function from A onto B.
- Let A be the set of integers and let B be the set of all equivalence classes corresponding to the relation R on $\mathbb{Z}$ defined as follows

$$aRb \text{ iff } a-b \text{ is a multiple of } 5$$

  Now define the function f from A to B as follows:

$$f(x)=[x],\ \forall x \in Z,$$

  where $[x]$ is the equivalence class of x (as it's defined by the relation R.) Is f one-to-one? Is it onto? Explain.
- Prove or disprove: If $f: A \longrightarrow B$ and $g: B \longrightarrow C$ are one-to-one and onto, then so is gof.
- Is it possible to find a bijective function from the interval $(0,1)$ onto the interval $(0,2)$? If yes, give an example.
- Is it possible to find a bijective function from the interval $(0,1)$ onto the interval $[0,2)$? If yes, give an example.
- Is it possible to find a bijective function from the interval $(0,1)$ onto the interval $[0,2]$? If yes, give an example.
- Is it possible to find a bijective function from $\mathbb{R}^+$ onto the interval $(0,\frac{1}{3})$? If yes, give an example.
- Is it possible to find a bijective function from $\mathbb{R}$ onto the interval $(0,\frac{1}{3})$? If yes, give an example.
- Is it possible to find a bijective function from $(\mathbb{N}\cup\{0\})\times(\mathbb{N}\cup\{0\})$ onto $\mathbb{Z}$? If yes, give an example.
- Prove or disprove the following: Let $f: A \longrightarrow B$ and $g: B \longrightarrow C$ be functions, let $\phi \neq S \subseteq A$ and $\phi \neq T \subseteq B$, let G be preimage of the image of S and H the image of the preimage of T, and let a, b, c, and d be real numbers and let $a<b$ and $c<d$. (For the first three parts, if the statement is false and if one direction is true, indicate that, and give a counterexample to show the other direction is false.) Then

(1) $g \circ f$ is one-to-one iff g and f are one-to-one.

(2) $g \circ f$ is onto iff g and f are onto.

(3) $g \circ f$ is bijection iff g and f are bijection.

(4) $G = S$. If false, indicate if one of these two sets is a subset of the other. Does your answer change if f is one-to-one?

(5) $H = T$. If false, indicate if one of these two sets is a subset of the other. Does your answer change if f is onto?

- Give an example of a bijection from $\mathbb{R}$ to (a, ∞).
- Give an example of a bijection from $(0, 1)$ to (a, ∞).
- Find the inverse of $f : \mathbb{R} \longrightarrow (4, \infty)$, $f(x) = 4 + e^{3x-5}$ if it exists.

  Solution: The function is a bijection (a similar example was done in class). Hence, it's invertible. Now let's find the inverse. First, replace $f(x)$ by y.

  $y = 4 + e^{3x-5}$.

  $y - 4 = e^{3x-5}$.

  $ln(y - 4) = ln(e^{3x-5})$.

  $3x - 5 = ln(y - 4)$.

  $3x = 5 + ln(y - 4)$.

  $x = \frac{5+ln(y-4)}{3}$.

  Hence, $f^{-1}(x) = \frac{5+ln(x-4)}{3}$. Note that f^{-1} is a bijection from $(4, \infty)$ to $\mathbb{R}$. Note also $|\mathbb{R}| = |(4, \infty)|$, and hence, $(4, \infty)$ is uncountable.
- Find the inverse of $f : (\frac{5}{3}, \infty) \longrightarrow \mathbb{R}$, $f(x) = 4 + ln(3x - 5)$ if it exists.

  Solution: The function is a bijection (a similar example was done in class). Hence, it's invertible. Now let's find the inverse. First, replace $f(x)$ by y.

  $y = 4 + ln(3x - 5)$.

  $y - 4 = ln(3x - 5)$.

  $e^{y-4} = e^{ln(3x-5)}$.

  $3x - 5 = e^{y-4}$.

  $3x = 5 + e^{y-4}$.

  $x = \frac{5+e^{y-4}}{3}$.

Hence, $f^{-1}(x) = \frac{5+e^{x-4}}{3}$. Note that f^{-1} is a bijection from $\mathbb{R}$ to $(\frac{5}{3}, \infty)$. Note also $|\mathbb{R}| = (\frac{5}{3}, \infty)$, and hence, $(\frac{5}{3}, \infty)$ is uncountable.

- Find the inverse of $f : \mathbb{R} \longrightarrow \mathbb{R}$, $f(x) = 4 + (3x - 5)^7$ if it exists.

  Solution: The function is a bijection (a similar example was done in class). Hence, it's invertible. Now let's find the inverse. First, replace $f(x)$ by y.

  $y = 4 + (3x - 5)^7$.
  $y - 4 = (3x - 5)^7$.
  $(y - 4)^{\frac{1}{7}} = 3x - 5$.
  $3x = 5 + (y - 4)^{\frac{1}{7}}$.
  $x = \frac{5+(y-4)^{\frac{1}{7}}}{3}$.
  Hence, $f^{-1}(x) = \frac{5+(x-4)^{\frac{1}{7}}}{3}$. Note that f^{-1} is a bijection from $\mathbb{R}$ to $\mathbb{R}$.

More material will be added to this section in the near future.