Real Vector Spaces and Supspaces

Definition: A real vector space is a set V of elements (called vectors) together with two operations $\oplus$ (called vector addition) and $\odot$ (called scalar multiplication) satisfying the following 10 properties (axioms):

(α) If u and v are elements of V, then $u \oplus v$ is in V.
- (a) $u \oplus v = v \oplus u$, for all u and v in V.
- (b) $u \oplus (v \oplus w) = (u \oplus v) \oplus w$, for all u, v, and w in V.
- (c) There is an element 0 in V such that $u \oplus 0 = 0 \oplus u = u$, for all u in V.
- (d) For each u in V, there is an element $-u$ in V such that $u \oplus -u = 0$.

(β) If u is an element of V and $c \in \mathbb{R}$, then $c \odot u$ is in V.
- (e) $c \odot (u \oplus v) = c \odot u \oplus c \odot v$, for all $c \in \mathbb{R}$ and all u and v in V.
- (f) $(c + d) \odot u = c \odot u \oplus d \odot u$, for all c and d in $\mathbb{R}$ and all u in V.
- (g) $c \odot (d \odot u) = (cd) \odot u$, for all c and d in $\mathbb{R}$ and all u in V.
- (h) $1 \odot u = u$, for all u in V.

Remarks:

(1) If u is a vector in a vector space, then u is written as $\mathbf{u}$ or $\vec{u}$.
(2) If the real numbers (called scalars) in the axioms above are complex, the vector space in that case is called a complex vector space.
(3) The vector **-u** is called the negative of **u** and the vector **0** is called the zero vector.
(4) **-u** and **0** are unique.
(5) We will write sometimes $u \oplus v$ as $u + v$ and $c \odot u$ as cu.
(6) Vectors of $\mathbb{R}^n$ are sometimes written as column vectors and sometimes as $(x_1, x_2, \cdots, x_n)$.

Theorem: Let V be a vector space

(1) $0 \odot u = 0$, for every u in V.
(2) $c \odot 0 = 0$, for every real number c.

(3) If $c \odot u = 0$, then either $c = 0$ or $u = 0$ (c here is a real number and $u \in V$).

(4) $(-1) \odot u = -u$, for all $u \in V$.

Examples of Vector Spaces:

(1) $\mathbb{R}$ with addition of real numbers and multiplication of real numbers.

(2) $\mathbb{R}^n$, $\forall n \in \mathbb{Z}^+$ with addition of vectors and scalar multiplication.

(3) M_{mn}: The set of all $m \times n$ matrices with matrix addition and scalar multiplication.

(4) $F[a, b]$: The set of all real-valued functions defined on $[a, b]$, where $\oplus$ and $\odot$ are defined as follows: $(f \oplus g)(t) = f(t) + g(t)$, $(c \odot f)(t) = cf(t)$.

(5) $F(-\infty, \infty)$: The set of all real-valued functions defined on $(-\infty, \infty)$.

(6) $C[a, b]$: The set of all continuous real-valued functions defined on $[a, b]$, where $\oplus$ and $\odot$ are defined as follows: $(f \oplus g)(t) = f(t) + g(t)$, $(c \odot f)(t) = cf(t)$.

(7) $C(-\infty, \infty)$: The set of all continuous real-valued functions defined on $(-\infty, \infty)$.

(8) P_n: The set of all polynomials of degree $\leq n$ (that includes the zero polynomial), where $\oplus$ and $\odot$ are defined as follows: For $p(t) = a_0 + a_1 t + a_2 t^2 + \cdots + a_n t^n$ and $q(t) = b_0 + b_1 t + b_2 t^2 + \cdots + b_n t^n$ in P_n and $c \in \mathbb{R}$, $p(t) \oplus q(t) = (a_0 + b_0) + (a_1 + b_1)t + (a_2 + b_2)t^2 \cdots + (a_n + b_n)t^n$, and $c \odot p(t) = ca_0 + ca_1 t + ca_2 t^2 + \cdots + ca_n t^n$.

(9) P: The set of all polynomials (that includes the zero polynomial).

Remark: We'll be dealing only with real vector spaces (although the field $\mathbb{R}$ can be replaced by any field such as the field $\mathbb{C}$ of complex numbers). Thus, we'll assume all vector spaces below are real vector spaces (i.e. they are vector spaces over $\mathbb{R}$).

Definition: Let V be a vector space and let W be a nonempty subset of V. If W is a vector space with respect to the operations in V, then W is called a subspace of V.

Definition: Let V be a vector space and let W be a nonempty subset of V. Then W is a subspace of V if and only if

(α) For every u and v in W, $u \oplus v$ is in W.

(β) For every u in W and every real number c, $c \odot u$ is in W.

Remarks:

(1) (α) and (β) in the definition above can be replaced by

For every u and v in W, and every c and d in $\mathbb{R}$, $c \odot u \oplus d \odot v$ is in W.

(2) If the word "nonempty" in the previous example is deleted, then we should add the following item to (α) and (β) above:

(γ) $\qquad 0 \in W$.

(3) If V is a vector space, then V and $\{0\}$ are subspaces of V. $\{0\}$ is called the zero subspace.

(4) Every vector space must contain the zero vector.

(5) If a subset W of a vector space V does not contain the zero vector, then W is not a subspace of V. But, if a subset W of a vector space does contain the zero vector that does not necessarily means W is a subspace of V.

(6) Let A be an $m \times n$ matrix and let W be the set of all solutions to $Ax = 0$. Then, W is a subspace of $\mathbb{R}^n$ and it's called the *nullspace* of A or the *solution space* of the system $Ax = 0$.

(7) P_n is a subspace of P_{n+1} and P_n is a subspace of P.

(8) To check if a subset W of a vector space V is a subspace of V, you need first to check that W is nonempty. If W is nonempty, then check closure (with respect to W) of $\odot$ and $\oplus$. If W is not closed under $\odot$ or under $\oplus$, then it is not a subspace.

Definition: Let $v_1,\ v_2, \ldots, v_k$ be vectors in a vector space V. A vector v in V is called a *linear combination* of $v_1,\ v_2, \ldots, v_k$ if there exist real numbers $c_1,\ c_2,\ \cdots,\ c_n$, such that

$$v = c_1v_1 + c_2v_2 + \cdots + c_kv_k.$$

Remark: To check if a vector v is a linear combination of $v_1,\ v_2,\ \cdots,\ v_k$, check if the linear system $c_1v_1 + c_2v_2 + \cdots + c_kv_k = v$ has a solution (note that the unkwons here are $c_1,\ c_2,\ \cdots,\ c_k$). If yes, then v is a linear combination of the given vectors. If not, then not.

Definition: Let $S = \{v_1, v_2, \cdots, v_k\}$ be a set of vectors in a vector space V. Then the set of all vectors in V that are linear combinations of the vectors in S is called span S or span$\{v_1, v_2, \cdots, v_k\}$.

Theorem: Let $S = \{v_1, v_2, \cdots, v_k\}$ be a set of vectors in a vector space V. Then span S is a subspace of V.

Exercises:

(1) Determine whether the following sets V are closed under $\oplus$ and $\odot$
   (a) V is the set of all ordered pairs of real numbers (x, y), where $x > 0$ and $y > 0$; $(x, y) \oplus (u, v) = (x + u, y + v)$, $c \odot (x, y) = (cx, cy)$.
   (b) V is the set of all ordered triples of real numbers of the form $(0, x, y)$; $(0, x, y) \oplus (0, u, v) = (0, x + u, y + v)$, $c \odot (0, x, y) = (0, 0, cz)$.
   (c) V is the set of all polynomials of the form $at^2 + bt + c$, where a, b, and c, are real numbers with $b = a + 1$; $(a_1t^2 + b_1t + c_1) \oplus (a_2t^2 + b_2t + c_2) = (a_1 + a_2)t^2 + (b_1 + b_2)t + (c_1 + c_2)$, $r \odot (at^2 + bt + c) = rat^2 + rbt + rc$.

(2) Determine whether the given set V with the given oparations is a vector space:
   (a) V is the set of all ordered pairs of real numbers (x, y); $(x, y) \oplus (u, v) = (x + u, y + v)$, $c \odot (x, y) = (0, 0)$.
   (b) V is the set of all real numbers; $u \oplus v = 2u - v$, $c \odot u = cu$.
   (c) V is the set of all positive real numbers; $u \oplus v = uv$, $c \odot u = u^c$.

(3) Show that if $u \neq 0$ and $a \odot u = b \odot u$, then $a = b$.

(4) Which of the following subsets of $\mathbb{R}^3$ are subspaces of $\mathbb{R}^3$? The set W of all vectors of the form
   (a) (a, b, c), where $a = c = 0$.
   (b) (a, b, c), where $a = -c$.
   (c) (a, b, c), where $b = 2a + 1$.

(5) Which of the following subsets of $\mathbb{R}^2$ are subspaces of $\mathbb{R}^2$? The set W of all vectors of the form
   (a) The union of the first and the third quadrant.
   (b) The set of all points in the unit disk (on and inside the unit circle).

(6) Which of the following subsets of P_2 are subspaces of P_2? The set W of all polynomials of the form

(a) $a_0 + a_1 t + a_2 t^2$, where $a_0 = a_1 = 0$.

(b) $a_0 + a_1 t + a_2 t^2$, where $a_1 = 2a_0$.

(c) $a_0 + a_1 t + a_2 t^2$, where $a_0 + a_1 + a_2 = 2$.

(d) $a + t^2$.

(e) $a_0 + a_1 t + a_2 t^2$, where $a_0,\ a_1,\ a_2$ are integers.

(f) $p(t) = a_0 + a_1 t + a_2 t^2$, where $p(0) = 0$.

(7) Which of the following subsets of M_{23} are subspaces of M_{23}? The set W of all matrices of the form

(a) $\begin{bmatrix} a & b & c \\ d & 0 & 0 \end{bmatrix}$, where $b = a + c$.

(b) $\begin{bmatrix} a & b & c \\ d & 0 & 0 \end{bmatrix}$, where $c > 0$.

(c) $\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$, where $a = -2c$ and $f = 2e + d$.

(8) Which of the following subsets of M_{nn} are subspaces of M_{nn}?

(a) The set of all $n \times n$ symmetric matrices.

(b) The set of all $n \times n$ nonsingular matrices.

(c) The set of all $n \times n$ diagonal matrices.

(d) The set of all $n \times n$ singular matrices.

(e) The set of all $n \times n$ upper triangular matrices.

(f) The set of all $n \times n$ matrices whose determinant is 1.

(9) Which of the following subsets are subspaces of $C(-\infty, \infty)$

(a) All nonnegative functions.

(b) All constant functions.

(c) All functions f such that $f(0) = 0$.

(d) All functions f such that $f(0) = 5$.

(e) All differentiable functions.

(10) Let $S = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$. Determine if $u = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ belongs to span S.

(11) Let $S = \left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \\ 6 \end{bmatrix} \right\}$. Determine if $u = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$ belongs to span S (i.e. if it's a linear combination of the vectors of S). How many vectors are in span S?

(12) Let W be the set of all vectors of the form $\begin{bmatrix} s \\ 3s \\ 2s \end{bmatrix}$. Show that W is a subspace of $\mathbb{R}^3$ by finding a vector v in $\mathbb{R}^3$ such that $W = \text{span}\{v\}$.

(13) Let W be the set of all vectors of the form $\begin{bmatrix} 5a + 2b \\ a \\ b \end{bmatrix}$. Show that W is a subspace of $\mathbb{R}^3$ by finding vectors u and v in $\mathbb{R}^3$ such that $W = \text{span}\{u, v\}$.

(14) Let W be the set of all vectors of the form shown (a, b, and c are arbitrary real numbers). Find a set of vectors S that span W

(a) $\begin{bmatrix} 3a + b \\ 4 \\ a - 5b \end{bmatrix}$.

(b) $\begin{bmatrix} a - b \\ b - c \\ c - a \\ b \end{bmatrix}$.