Operations on Sets

Definition: A *set* is a collection of distinct unordered objects.

Note: The empty set is the set with no elements. The empty set is denoted by ϕ.

Definition: Let A, B, and C be subsets of a universal set U. Then

1. We say $A \subseteq B$ (pronounced A is a subset of B) if every element of A is an element of B.

2. We say $A \subset B$ (pronounced A is a proper subset of B) if every element of A is an element of B, but B has more elements than A.

3. We say $A = B$ (pronounced A is equal to B) if A and B have the same elements. Notice that $A = B$ iff $A \subseteq B$ and $B \subseteq A$.

4. $A \cup B$ (pronounced A union B) is the set of elements that are either in A or in B.

5. $A \cap B$ (pronounced A intersection B) is the set of elements that are in both A and B (i.e. elements common to A and B).

6. $A - B$ (pronounced A minus B) is the set of elements that are in A but not in B. Notice that $A - B = A - (A \cap B)$. Sometimes $A - B$ is written as $A \setminus B$.

7. $A \oplus B$ (called the *symmetric difference* of A and B) is the set of elements that are in A but not in B and the elements that are in B but not in A. Notice that $A \oplus B = (A - B) \cup (B - A) = (A \cup B) - (A \cap B)$.

8. $\overline{A}$ (pronounced the complement of A) is the set of elements that are in U but not in A. Notice that $\overline{A} = U - A$. Sometimes $\overline{A}$ is written as A^c.

9. $A \times B$ (the cross product of A and B) is the set $\{(a, b) \mid a \in A, b \in B\}$. Be careful $A \times B$ is not necessarily equal to $B \times A$.

10. $A \times B \times C$ (the cross product of A, B, and C) is the set $\{(a, b, c) \mid a \in A, b \in B, c \in C\}$.

11. $\mathcal{P}(A)$ (called the *power set* of A) is the set of all subsets of A. Notice that ϕ is a subset of any set.

Facts About Sets

Let A, B, and C be subsets of a universal set U. Then

1. $A \subseteq A$, $A = A$, $A \cup A = A$, $A \cap A = A$, $A - A = \phi$.
2. $\overline{\overline{A}} = A$.
3. $A \cup \overline{A} = U$, $A \cap \overline{A} = \phi$.
4. $A \cap U = A$, $A \cup U = U$.
5. not$(x \in A)$ is equivalent to $x \notin A$ is equivalent to $x \in \overline{A}$.
6. $\overline{\phi} = U$, $\overline{U} = \phi$.
7. $A - B = A \cap \overline{B}$.
8. $\overline{A \cap B} = \overline{A} \cup \overline{B}$, $\overline{A \cup B} = \overline{A} \cap \overline{B}$.
9. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$, $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.
10. $A \cap (B \cap C) = (A \cap B) \cap C$, $A \cup (B \cup C) = (A \cup B) \cup C$.
11. $A \cup B = B \cup A$, $A \cap B = B \cap A$.
12. $A \cap B \subseteq A$, $A \cap B \subseteq B$, $A \subseteq A \cup B$, $B \subseteq A \cup B$.
13. $A \cup \phi = A$, $A \cap \phi = \phi$, $\phi \subseteq A$.

Remark: $A \times A$ is usually written as A^2. Similarly, $A \times A \times A$ is written as A^3, and so on.

Questions

1. Decide whether the following are true or false. Explain your answers.
   (a) If $A \subseteq B$, then $\overline{A} \subseteq \overline{B}$.
   (b) If a and b are irrational numbers, then a^b is irrational.
   (c) If a and b are irrational numbers and $|a| \neq |b|$, then $a + b$ is irrational.
   (d) If n is an odd integer, then $n^2 - 4n - 3$ is an even integer.
   (e) $\phi \in \{2, 3\}$.

(f) $\phi \subseteq \{2, 3\}$.

(g) $\{2\} \in \{2, 3\}$.

(h) $\{2\} \subseteq \{2, 3\}$.

(i) $\phi \in \{\phi, 2, 3\}$.

(j) $\phi \subseteq \{\phi, 2, 3\}$.

(k) $\{2\} \in \{\{2\}, 2, 3\}$.

(l) $\{2\} \subseteq \{\{2\}, 2, 3\}$.

2. Let $A = (-2, 4] \cup [6, 10)$, $B = [-5, 1) \cup (8, 12)$, $C = (3, 5] \cup \{7\}$, $D = [-1, 1]$.. Find $A \cap B$, $A \cup B$, $A - B$, $A \oplus B$, $B - A$. Then describe $C \times C$ and $D \times D$.

3. $1 + 3\mathbb{Z}^+ \subset 1 + 6\mathbb{Z}^+$.