CSCE 235 **Feb 14, 01**

Binary Relations

Definition 1. Let R_1 be a binary relation from X to Y and R_2 a binary relation from Y to Z; then

$$R_2 o R_1 = \{(x, z) \mid (x, y) \in R_1 \text{ and } (y, z) \in R_2 \text{ for some } y \in Y\}.$$

Definition 2. Let R be a relation on X; then

$$R^{-1} = \{(x, y) \mid (y, x) \in R\}.$$

Question: Let R_1 be a binary relation from X to Y and R_2 a binary relation from Y to Z. Prove or disprove: $R_1 o R_2 = R_2 o R_1$.

Solution: The satement is false. You can come up with a counterexample easily. Do not forget that

$R_1 o R_2$ may be even undefined.

Remark 0.1. Notice that every function is a relation. In other words, functions are special cases of relations. For example, if f is the function from $X = \{-1, 0, 2\}$ onto $Y = \{0, 1, 4\}$ given by $f(x) = x^2$. Then f can be represented as $f = \{(-1, 1), (0, 0), (4, 4)\}$.

Notice also that if f is a function from X into X and if f^{-1} exists, then $fof^{-1}(x) = f^{-1}of(x) = x$, $\forall x \in X$. As a relation, we have $fof^{-1} = f^{-1}of = \{(x, x) \mid x \in X\}$. The question now is: does the same thing hold for any relation R on a set X? If no, then what are the properties of RoR^{-1}? We'll address some of those properties shortly.

Question: Let R be a relation on X. Prove or disprove:

(1) $RoR^{-1} = R^{-1}oR$.

(2) $|RoR^{-1}| = |R^{-1}oR|$.

(3) RoR^{-1} and $R^{-1}oR$ are both symmetric.

(4) If R is symmetric, then $R^{-1} = R$.

(5) $RoR = R$.

Solution:

(1) False. Counterexample: take $X = \{1, 2, 3\}$ and $R = \{(1, 3)\}$. Then $R^{-1} = \{(3, 1)\}$. Hence, $RoR^{-1} = \{(3, 3)\}$ and $R^{-1}oR = \{(1, 1)\}$.

(2) False. Counterexample: take $X = \{1, 2, 3\}$ and $R = \{(1, 2), (3, 2)\}$. Then $R^{-1} = \{(2, 1), (2, 3)\}$. Hence, $RoR^{-1} = \{(2, 2)\}$ and $R^{-1}oR = \{(1, 1), (1, 3), (3, 1), (3, 3)\}$.

(3) True. The proof was done in class.

(4) Straightforward.

(5) Easy.

Question: Let R be a relation on X. Prove or disprove:

(1) RoR^{-1} is transitive.

(2) RoR^{-1} is reflexive.

Question: Let R be the relation on $\mathbb{Z}$ defined by:

$$aRb \text{ iff } 3a + b \text{ is a multiple of} 4.$$

Is R an equivalence relation on $\mathbb{Z}$? If yes, then what is [1]. Also, find R^{-1}.

Solution: R is an equivalence relation, because

(1) R is reflexive, because $\forall a \in Z$, $3a + a = 4a$ which is divisible by 4. Thus, $(a, a) \in \mathbb{R}$.

(2) R is symmetric, because if $(a, b) \in R$, then $3a + b = 4m$, for some $m \in \mathbb{Z}$. Thus, $b = 4m - 3a$. But now $3b + a = 3(4m - 3a) + a = 4(3m - 2a)$. Notice that $3m - 2a$ is an integer. Thus, $3b + a$ is divisible by 4, which means that $(b, a) \in R$.

(3) R is transitive, because if (a, b) and (b, c) are in R, then $3a+b = 4m$ for some $m \in \mathbb{Z}$ and $3b + c = 4k$ for some $k \in \mathbb{Z}$. Therefore, $3a + c = 4m - b + 4k - 3b = 4(m + k - b)$. Notice that $m + k - b$ is an integer. Thus, $3a + c$ is divisible by 4, which means $(a, c) \in R$.

$$\begin{aligned}
[1] &= \{b \in \mathbb{Z} \mid aR1\} \\
&= \{a \in \mathbb{Z} \mid 3a + 1 = 4k,\ k \in \mathbb{Z}\} \\
&= \{a \in \mathbb{Z} \mid a = \frac{4k-1}{3},\ k \in \mathbb{Z}\} \\
&= \{a \in \mathbb{Z} \mid a = 4k - 3,\ k \in \mathbb{Z}\}
\end{aligned}$$

$R^{-1} = \{(a, b) \mid a + 3b \text{ is a multiple of } 4\}$.

Question: Define the following relation on R^2:

$$(a, b)R(c, d) \text{ iff } a^2 + b^2 = c^2 + d^2.$$

Is R an equivalence relation? If yes, then what is the equivalence class of (α, β) and what does it represent in the Cartesian plane? Also, find R^{-1}.

Solution: R is an equivalnce relation. This can be easily verified. Also, it is easy to see that $R^{-1} = R$. Why?

Now, $[(\alpha, \beta)] = \{(x, y) \mid x^2 + y^2 = \alpha^2 + \beta^2\}$. It is easy to see that $[(\alpha, \beta)]$ is a circle centered at the origin and with readius $\sqrt{\alpha^2 + \beta^2}$. Actually, the disjoint equivalence classes partition the Cartesian plane (i.e. R^2) into a set of circles centered at the origin.