Matrices Definitions and Facts

If A and B are matrices, then $A+B$ is defined only if A and B have the same size. If A and B have the same size, then the element of $A+B$ in row i column j is equal to the element of A in row i column j plus the element of B in row i column j. I.e. $(A+B)(i,j) = A(i,j) + B(i,j)$.

If A and B are m by n matrices, then $A+B = B+A$ and $A+(B+C) = (A+B)+C$.

If A is a matrix and r is a real number, then the matrix rA is defined as follows: The element of rA in row i column j is equal to r times the element of A in row i column j. I.e. $(rA)(i,j) = rA(i,j)$.

If $u = (u_1, u_2, \cdots, u_n)$ and $v = (v_1, v_2, \cdots, v_n)$ are vectors, then the dot product of u and v, denoted $u \cdot v$ is equal to $\sum_{i=1}^{n} u_i v_i$ and $u + v = (u_1 + v_1, u_2 + v_2, \cdots, u_n + v_n)$.

If A and B are matrices, then AB is defined iff the number of columns of A is equal to the number of rows of B. So, assume A is m by n and B is n by q, then the element of AB in row i column j is equal to the dot product of row i of A with column j of B. I.e. $(AB)(i,j) = A(i,*) \cdot B(*,j) = \sum_{k=1}^{n} A(i,k)B(k,j)$.

Notice that if AB is defined, then it is not necessarily for BA to be defined . Even if BA is defined, then it's not necessarily that $AB = BA$.

If A and B are matrices such that all following additions and multiplications are defined, then $A(BC) = (AB)C$ and $A(B+C) = AB + AC$.

Definition: Let A be a square matrix of dimension n (i.e. A is n by n). If there exists a n by n matrix B such that $AB = I$. Then

1. $BA = I$.
2. A is said to be invertible and nonsingular.
3. B is called the inverse of A and A is called the inverse of B.
4. The inverse of A is denoted by A^{-1}.

Thus, if A and B are matrices such that $AB = I$, then A and B are both invertible and $A^{-1} = B$, $B^{-1} = A$.

Notice also that if A is invertible (nonsingular), then there exist a matrix B such that $AB = I$.

Facts and Remarks:

1. The inverse is defined only for square matrices. But, the inverse of a square matrix may not exist.

2. Let
$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$
Then the determinant of A is defined to be the quantity $ad - bc$.

3. If A is a 2 by 2 matrix, then A^{-1} exists iff the determinant of A is nonzero.

4. If
$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$
and if $ad - bc \neq 0$. Then
$$A^{-1} = \frac{1}{D}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix},$$
where D is the determinant of A.

5. If k is a positive integer greater than 1 and A is a square matrix, then $A^k = AA^{k-1}$, or equivalently $A^k = A^{k-1}A$. If k and h are positive integers, then $A^kA^h = A^{k+h}$, $(A^k)^h = A^{kh}$, and $A^0 = I$.

6. If k is a positive integer greater than 1 and if A is a square invertible matrix, then $A^{-k} = (A^k)^{-1} = (A^{-1})^k$.

7. If A and B are inevrtible n by n matrices, then AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$. Also, BA is invertible and $(BA)^{-1} = A^{-1}B^{-1}$.

8. If A is an invertible n by n matrix and if r is a nonzero real number, then both rA and A^{-1} are invertible and $(rA)^{-1} = \frac{1}{r}A^{-1}$ and $(A^{-1})^{-1} = A$.

9. If A and B are invertible n by n matrices, then it is not necessarily for $A+B$ to be invertible and even if $A+B$ is invertible, then in general, $(A+B)^{-1} \neq A^{-1}+B^{-1}$.

10. $I^{-1} = I$. Notice that $II = I$.

Definition: Let A be an m by n matrix, then the *transpose* of A, denoted A^T, is defined as follows: $A^T(i,j) = A(j,i)$, for all $1 \leq i \leq m$ and $1 \leq j \leq n$.

In other words, the element of A^T in row i column j is equal to the element of A in row j column i. Thus, to get A^T from A, all you have to do is to change the rows of A to columns. In other words, $A^T(*,i) = A(i,*)$. Notice that A^T is n by m.

Note: $(A^T)^T = A$.

Facts: Let A and B be n by n matrices and let k be a positive integer greater than 1. Then

1. $(AB)^T = B^T A^T$, $(BA)^T = A^T B^T$, and $(A+B)^T = A^T + B^T$.
2. $(A^k)^T = (A^T)^k$.
3. If A is invertible, then so is A^T and $(A^T)^{-1} = (A^{-1})^T$. Thus, if A is invertible, then so is $(A^k)^T$ and $((A^k)^T)^{-1} = ((A^{-1})^k)^T = (A^{-k})^T$.

Definition: A matrix A is said to be *symmetric* if $A^T = A$.

Definition: A matrix A is said to be *skew-symmetric* if $A^T = -A$.

Definition: A matrix A is said to be *orthogonal* if $A^T = A^{-1}$.