Binomial Theorem

Notation:

Let n and r be nonnegative integers and assume $r \leq n$. We will use $C(n, r)$ to denote the quantity $\frac{n!}{r!(n-r)!}$.

Note: $0! = 1$ by definition.

Example 1. Let n and r be natural numbers such that $r \leq n$. It is easy to prove (using the definition) that:

- $C(n, n) = C(n, 0) = 1$.
- $C(n, 1) = C(n, n-1) = n$.
- $C(n, r) = C(n, n-r)$.

Theorem 2. *Let a and b be real numbers and let $n \in \mathbb{N}$. Then*

$$(a+b)^n = \sum_{k=0}^{n} C(n, k) a^{n-k} b^k.$$

Remarks:

- The above expansion has $n+1$ terms.
- $(a+b)^n = \sum_{k=0}^{n} C(n, k) a^k b^{n-k}$.

Example 3. Prove that if r is a nonnegative integer and $n \in \mathbb{N}$, then

$$r^n = \sum_{k=0}^{n} C(n, k)(r-1)^k.$$

Solution: Done in class.

Example 4. Prove that $\sum_{k=0}^{n} C(n, k) = 2^n$, $\forall n \in \mathbb{N}$.

Solution:

A direction application of the previous example.

Example 5. Prove that $\sum_{k=0}^{n} C(n,k)(-1)^k = 0$, $\forall n \in \mathbb{N}$.

Solution:

A direction application of example 3.

Example 6. Prove that $\sum_{k=0}^{n} C(n,k)(2)^k = 3^n$, $\forall n \in \mathbb{N}$.

Solution:

A direction application of example 3.

Example 7. Prove that $\sum_{k=0}^{n/2} C(n,2k) = \sum_{k=1}^{n/2} C(n,2k-1) = 2^{n-1}$, $\forall n \in \mathbb{N}$.

Solution:

From example 4, we have

$$2^n = \sum_{k=0}^{n} C(n,k) = \sum_{k=0}^{n/2} C(n,2k) + \sum_{k=1}^{n/2} C(n,2k-1).$$

Thus,

$$\sum_{k=0}^{n/2} C(n,2k) + \sum_{k=1}^{n/2} C(n,2k-1) = 2^n. \tag{0.1}$$

But, from example 5, we have

$$\sum_{k=0}^{n/2} C(n,2k) - \sum_{k=1}^{n/2} C(n,2k-1) = 0.$$

Hence,

$$\sum_{k=0}^{n/2} C(n,2k) = \sum_{k=1}^{n/2} C(n,2k-1).$$

Thus, equation 0.1, becomes

$$\sum_{k=0}^{n/2} C(n,2k) + \sum_{k=0}^{n/2} C(n,2k) = 2^n.$$

Therefore,

$$2\sum_{k=0}^{n/2} C(n, 2k) = 2^n.$$

The result now follows.

EXAMPLE 8. Find the coefficients of x^{11} and x^{13} in the binomial expansion of $(\frac{x^4-2}{x})^{17}$. Also, find the two middle terms.

Solution: In class.

EXAMPLE 9. Find the coefficient of $x^{11}y^6$ in the binomial expansion of $(2x - y)^{17}$. Also, find the two middle terms.

Solution: In class.

EXAMPLE 10. Find the coefficient of x^{18} in the binomial expansion of $(2x - y)^{18}$. Also, find the middle term.

Solution: In class.