

CSIT 241 - FINAL EXAM

Name:

Instructions: Do all of the following. EXPLAIN every step. Points will be deducted for incomplete proofs or incomplete solutions. Do not look at your neighbor or talk to him. If you use a method different than what the question is asking for, you'll get no credit. Use only notation used in class. Any violation of the instructions may result in a partial credit or no credit at all.

Other instructions: The exam is closed book, closed notes. No material can be used and the Internet is not allowed. Do not use calculators.

Time: 2 hours.

- (1) (5 points) Find the negation and the contrapositive (indicate which is which) of the following:

If x is even or y is even, then xy is even.

- (2) (10 points) The following statements are true. Prove them.

(a) The set $4\mathbb{Z}^-$ has no smallest element.

(b) If x is irrational, then $\sqrt{x}$ is irrational.

- (3) (5 points) The following statement is false. Give a counterexample to prove it is false.

If a and b are irrational, then a^b is irrational.

(4) (8 points) Prove the following by mathematical induction.

$$1 + 5 + 5^2 + \cdots + 5^n = \frac{-1+5^{(n+1)}}{4}, \forall n \geq 1.$$

(5) (9 points)

(a) Let $A = \mathbb{Z}$, $\mathcal{R} = \{(x, y) \in A \times A \mid 3x + 2y + 4 > 2x + 3y + 7\}$. Prove by a counterexample that $\mathcal{R}$ is not reflexive.

(b) Let $A = \mathbb{Z}$, $\mathcal{R} = \{(x, y) \in A \times A \mid x - y > 3\}$. Prove that $\mathcal{R}$ is antisymmetric.

(c) Let $A = \mathbb{Z} - \{0\}$, $\mathcal{R} = \{(x, y) \in A \times A \mid \gcd(x, y) = 1\}$. Prove by a counterexample that $\mathcal{R}$ is not transitive.

(d) Let $\mathcal{R} = \{(x, y) \in \mathbb{Z} \times \mathbb{Z} \mid x - y = 2k, k \in \mathbb{Z}\}$. Find [7].

(6) (8 points) Prove by counterexamples that $f : \mathbb{R} \longrightarrow \mathbb{R}$,

$f(x) = 100 - (x - 20)^2$ is not onto and not one-to-one.

- (7) (8 points) Find the inverse of the function $f : \mathbb{R}^+ \longrightarrow (1, e^2)$, defined by $f(x) = e^{\left(\frac{2x}{x+3}\right)}$.

- (8) (12 points) Let $a = 159$ and $b = 50$. Find $\gcd(a, b)$ and write it as a linear combination of a and b . Also, find 50^{-1} in $\mathbb{Z}_{159}$ and $-200 \pmod{159}$.

- (9) (9 points)

- (a) How many permutations of the word **CONNECTING** are there if no two N's are to be consecutive?

Answer:

- (b) In how many ways can you arrange 70 distinct CS books, 20 distinct Physics books, and 30 distinct Math books on a shelf if the CS books are to be together and the Physics books are to be together?

Answer:

- (c) In how many ways can you reorder the letters of the word

ABCDEFGHJKLMNO so that exactly 7 of them remain in their original positions? Write the answer in terms of derangements.

Answer:

- (d) In how many ways can you distribute 15 distinct red balls into 25 distinct boxes at most one ball to a box?

Answer:

- (e) In how many ways can you distribute 15 distinct red balls into 25 distinct boxes? (Here there is no limit on the number of balls in each box.)

Answer:

- (f) In how many ways can you distribute 15 identical red balls into 25 distinct boxes at most one ball to a box?

Answer:

- (g) In how many ways can you distribute 15 identical red balls into 25 distinct boxes? (Here there is no limit on the number of balls in each box.)

- (h) In how many ways can you distribute 15 (distinct) red balls and 25 (identical) blue balls into 70 distinct boxes at most one ball to a box?

Answer:

- (i) In how many ways can you distribute 15 (distinct) red balls and 25 (identical) blue balls into 70 distinct boxes? (Here there is no limit on the number of balls in each box.) **Answer:**

Answer:

(10) Let

$$A = \begin{bmatrix} 1 & 3 & 2 \\ 4 & 0 & 2 \\ 3 & 4 & 3 \end{bmatrix}, B = \begin{bmatrix} 2 & 5 \\ 3 & 1 \end{bmatrix}.$$

(a) (5 points) Find $(B^2)^T$.

(b) (8 points) Find A^{-1} if exists.

- (c) (5 points) Use A^{-1} to find the solution of the system (do not use substitution or elimination):

$$x_1 + 2x_2 + 3x_3 = 4.$$

$$4x_1 + 5x_2 + 7x_3 = 1.$$

$$3x_1 + 6x_2 = 2.$$

- (11) (4 points) Find the coefficient of $x^{11}y^{12}$ and the coefficient of x^7y^8 in the expansion of $(2x - 3y)^{23}$.

- (12) (4 points) Prove or disprove that $L : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, given by $L(x_1, x_2) = (-x_2, 3)$ is a linear transformation.