CSIT 241 Cardinality Spring 2003

Facts

- Let $f: A \longrightarrow B$ and $g: B \longrightarrow C$ be functions.

1. If f and g are invertible, then so is gof and $(gof)^{-1} = f^{-1}og^{-1}$.
2. If f and g are 1-to-1, then so is gof. The converse is not necessarily true.
3. If f and g are onto, then so is gof. The converse is not necessarily true.
4. If f and g are bijections, then so is gof. The converse is not necessarily true.
5. If gof is 1-to-1, then f is 1-to-1.
6. If gof is onto, then g is onto.

- Let $f: A \longrightarrow B$ be an invertible function, then $fof^{-1} = 1_B$ and $f^{-1}of = 1_A$.

Cardinality

Definition: If A is a finite set, then the *cardinality* of A, denoted $|A|$, is the number of elements of A.

Definition: Let A and B be sets. We say A and B have the same cardinality, written $|A| = |B|$, iff there is a bijection between A and B.

Definition: A set A is *countably infinte* iff $|A| = |\mathbb{N}|$ and *countable* iff it is either finite or countably infinite. A set which is not countable is called *uncountable.*

Notation: The cardinality of $\mathbb{N}$; i.e. $|\mathbb{N}|$, is denoted by $\aleph_0$.

Facts

1. The following sets are countable: $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$.
2. The following sets are uncountable: $\mathbb{R}$, $\mathbb{R} - \mathbb{Q}$, $(0, 1)$.
3. A subset of a countable set is countable.

4. If A and B are countable, then $A \cup B$ countable.

Theorem: $|\mathbb{Z}| = |\mathbb{N}|$.

Proof: We have to present a bijection from $\mathbb{Z}$ to $\mathbb{N}$ or from $\mathbb{N}$ to $\mathbb{Z}$, because remember two sets have the same cardinality iff there is a bijection between them. The bijection that we'll present is from $\mathbb{N}$ to $\mathbb{Z}$. The bijection simply maps the set of even natural numbers to the whole set of natural numbers and the set of odd natural numbers to zero and the set of negative integers. In other words, our function is

$$f(x) = \begin{cases} \frac{x}{2} & \text{if } x \text{ is an even natural number} \\ \frac{1-x}{2} & \text{if } x \text{ is an odd natural number} \end{cases}$$

Notice that f is a function from $\mathbb{N}$ to $\mathbb{Z}$. Now we have to prove that f is one-to-one and onto. First, we prove it is one-to-one. So, assume $f(x_1) = f(x_2)$, where x_1 and x_2 are natural numbers. Now we have the following 4 possibilities:

1. x_1 and x_2 are both even. In this case, $f(x_1) = f(x_2)$ implies $\frac{x_1}{2} = \frac{x_2}{2}$, which implies $x_1 = x_2$.

2. x_1 and x_2 are both odd. In this case, $f(x_1) = f(x_2)$ implies $\frac{1-x_1}{2} = \frac{1-x_2}{2}$, which implies $x_1 = x_2$.

3. x_1 is even and x_2 is odd. In this case, $f(x_1) = f(x_2)$ implies $\frac{x_1}{2} = \frac{1-x_2}{2}$, which implies $x_1 + x_2 = 1$. But, x_1 and x_2 are both natural numbers. That means $x_1 \geq 1$ and $x_2 \geq 1$. Thus, $x_1 + x_2 \geq 2$. That means $x_1 + x_2$ cannot be equal to 1. Thus, if x_1 is even and x_2 is odd, then $f(x_1)$ cannot be equal to $f(x_2)$. Remember our assumption that $f(x_1) = f(x_2)$ led to a contradiction (which is $x_1 + x_2 = 1$) . That means the assumption is false. I.e. if x_1 is even and x_2 is odd, then $f(x_1)$ cannot be equal to $f(x_2)$.

4. x_1 is odd and x_2 is even. This case is similar to the previous case.

Now we'll prove f is onto. To do that, we'll have to prove that if $b \in \mathbb{Z}$, then there exists $a \in \mathbb{N}$ such that $f(a) = b$. Notice that f maps the set of even natural numbers to the set of natural numbers and it maps the set of odd natural numbers to $\mathbb{Z}^- \cup \{0\}$. Now let $b \in \mathbb{Z}$. If $b \in \mathbb{N}$, the preimage of b is $2b$. I.e. $f(2b) = b$. Notice that $2b$ is an even natural number, so we use the first rule (i.e. $f(x) = x/2$). On the other hand, if $b \in \mathbb{Z}^- \cup \{0\}$, the preimage of b is $1 - 2b$. I.e. $f(1 - 2b) = b$. Notice that $1 - 2b$ is an odd natural number, so we use the second rule (i.e. $f(x) = (1 - x)/2$). Now you may wonder how I found those preimages. You'll understand how if you find the inverse (although we haven't proved yet that f is onto, which means we can't conclude at this stage that f is bijection, so we can't talk about the inverse, but it's a good idea to try to find it). Notice first that f^{-1} is a function from $\mathbb{Z}$ to $\mathbb{N}$. Notice also that the first part of f maps the set of even natural numbers to the set of natural numbers. So, the first part of f^{-1} must map the set of natural numbers to the set of even natural numbers. This means we have to work now with the first rule of f (i.e. $f(x) = \frac{x}{2}$). So, assume $b \in \mathbb{N}$, then we want to find an even natural number a such that $f(a) = b$. I.e. we want $\frac{a}{2} = b$. Solve for a (i.e. write a in terms of b) to get $a = 2b$. So, if b is a natural number, then the preimage of b is $2b$ (notice that $2b$ is an even natural number). Now notice that the second part of f maps the set of odd natural numbers to the set of negative integers and zero. So, the second part of f^{-1} must map the set of negative integers and zero to the set of odd natural numbers. This means we have to work now with the second rule of f (i.e. $f(x) = \frac{1-x}{2}$). So, assume $b \in \{0\} \cup \mathbb{Z}^-$, then we want to find an odd natural number a such that $f(a) = b$. I.e. we want $\frac{1-a}{2} = b$. Solve for a (i.e. write a in terms of b) to get $a = 1 - 2b$. So, if b is a negative integer or zero, then the preimage of b is $1 - 2b$ (notice that $1 - 2b$ is an odd natural number).

Example: Try to find f^{-1} using the procedure which we used to prove f is onto. You'll find

$$f^{-1}(x) = \begin{cases} \text{2x} & \text{if } x \text{ is a natural number} \\ \text{1-2x} & \text{if } x \text{ is a negative integer or zero} \end{cases}$$

Notice that f^{-1} is a function from $\mathbb{Z}$ to $\mathbb{N}$. Notice also that f^{-1} is a bijection because f is a bijection.

Facts: $|\mathbb{Q}| = |\mathbb{N}|$, $|\mathbb{R}| = |(0,1)|$, $|\mathbb{R}| \neq |\mathbb{N}|$.

Definition: A set is said to be *countable* if it is either finite or countably infinite.

Exercises

1. Give an example of a bijection from $\mathbb{R}$ to $(5, \infty)$.
2. Give an example of a bijection from $\mathbb{N}$ to $\mathbb{N} \cup \{-1, -2\}$.
3. Give an example of a bijection from $[5, 8]$ to $[2, 4]$.