

Basis

Definition: The dimension of a nonzero vector space V , denoted $\dim V$, is the number of vectors in a basis for V . The vector space $\{0\}$ has dimension zero. A vector space of dimension n is called n -dimensional.

- (1) If V is a vector space of dimension $n > 0$. Then
 - (a) Any set of n linearly independent vectors spans V , and hence, it's a basis for V .
 - (b) Any n vectors that span V are linearly independent, and hence, it's a basis for V .
 - (c) If $m > n$, then any collection of m vectors in V is linearly dependent.
 - (d) No set of less than n vectors can span V .
 - (e) If $S = \{v_1, v_2, \dots, v_n\}$ is a basis for V and $T = \{w_1, w_2, \dots, w_k\}$ is linearly independent, then $k \leq n$.
 - (f) If $S = \{v_1, v_2, \dots, v_n\}$ is a basis for V and $T = \{w_1, w_2, \dots, w_k\}$ are bases for V , then $k = n$.
 - (g) Thus, given a set S of m vectors in V . If $m > n$, then S is linearly dependent and hence not a basis for V . If $m < n$, then S does not span V and hence not a basis for V . If $m = n$, then it suffices to check if S is linearly independent. If yes, then it's a basis for V . If not, then it is not a basis.
 - (h) Any subset of less than n linearly independent vectors can be extended to form a basis for V . Here is how to do that if the vector space is $\mathbb{R}^n$ or P_n : Suppose that you want to extend $\{v_1, \dots, v_k\}$ where $k < n$ to a basis for V and let $b_1, b_2, \dots, b_n$ be a basis for V . Form the matrix in which the first k columns are $v_1, \dots, v_k$ and make the remaining columns $b_1, b_2, \dots, b_n$. Now perform Gauss-Jordan elimination on the matrix to reduce it to reduced row echelon form, then the vectors corresponding to the columns with the leading 1's are the basis you're looking for.
- (2) Let $S = \{v_1, \dots, v_m\}$ be a set in a vector space V . Then

- (a) If one of the vectors in S , say v_j , is a linear combination of the remaining vectors in S , then $\text{span } S - \{v_j\} = \text{span } S$.
- (b) If $\text{Span } S \neq \{0\}$. Then some subset of S is a basis for $\text{span } S$. See how in the attached sheet.

Remarks:

- (1) A subspace has to contain zero. Thus, any subspace of $\mathbb{R}^2$ must pass through the origin. Trivial subspaces of $\mathbb{R}^2$ are $\{0\}$ (dimension 0) and itself (dimension 2). One-dimensional subspaces of $\mathbb{R}^2$ are lines through the origin.
- (2) The subspaces of $\mathbb{R}^3$ are itself (dimension 3), $\{0\}$ (dimension 0), all lines through the origin (dimension 1), and all planes through the origin (dimension 2).