Span and Linear Independence

Definition: The vectors $v_1,\ v_2,\ \cdots, v_n$ in a vector space V are said to span V if every vector in V is a linear combination of $v_1,\ v_2,\ \cdots, v_n$. If $S = \{v_1,\ v_2,\ \cdots, v_n\}$, we say S spans V, or V is spanned by S, or Span $S = V$.

Remark: To check if $v_1,\ v_2,\ \cdots, v_n$ span a vector space V, do the following

(1) Choose an arbitrary vector v in V.
(2) If v is a linear combination of the given vectors, then the given vectors span V. If not, then they do not span V.

Definition: The vectors $v_1,\ v_2,\ \cdots, v_n$ in a vector space V are said to be linearly dependent if there exists constans $c_1,\ c_2,\ \cdots, c_n$ not all zero such that $c_1v_1 + c_2v_2 + \cdots + c_nv_n = 0$. If $v_1,\ v_2,\ \cdots, v_n$ are not lineary dependent, then they are called linearly independent. Thus, $v_1,\ v_2,\ \cdots, v_n$, are lineary independent if and only if whenever $c_1v_1 + c_2v_2 + \cdots + c_nv_n = 0$, then $c_1 = c_2 = \cdots = c_n = 0$. If $S = \{v_1,\ v_2,\ \cdots, v_n\}$ and $v_1,\ v_2,\ \cdots, v_n$ are linearly dependent, we say S is linearly dependent and if $v_1,\ v_2,\ \cdots, v_n$ are linearly independent, we say S is linearly independent.

Remark: To check if $v_1,\ v_2,\ \cdots, v_n$ are linearly independent/dependent, do the following

(1) Form the equation $c_1v_1 + c_2v_2 + \cdots + c_nv_n = 0$.
(2) If the homogeneous system obtained has only the trivial solution, then the vectors are linearly independent; if it has a nontrivial solution, then the vectors are linearly dependent.

Definition: The nullspace of an $m \times n$ matrix A (or of $Ax = 0$), denoted Nul A, is the set of all solutions of $Ax = 0$.

Theorem:

(1) The nullspace of an $m \times n$ matrix A is a subspace of R^n.

(2) Nul $A = \{0\}$ iff $Ax = 0$ has only the trivial solution.

(3) If the number of equations in a homogeneous linear system is more than the number of unkowns/variables, then the system has infinitely many solutions.

(4) If A is an $n \times n$ singular matrix, then $Ax = 0$ has infinitely many solutions; and if it's nonsingular, then $Ax = 0$ has only the trivial solution.

(5) If a finite set S of vectors contain the zero vector, then S is linearly dependent.

(6) The nonzero vectors $v_1, \ v_2, \cdots, v_n$ in a vector space V are linearly dependent iff one of the vectors, $v_j, \ j \geq 2$, is a linear combination of the proceeding vectors $v_1, \ v_2, \cdots, v_{j-1}$.

(7) A finite set S of vectors in a vector space V is linearly dependent iff one of the vectors in S is a linear combination of all the other vectors in S.

(8) Let A and B be finite subsets of a vector space V, and let $A \subseteq B$. Then,
   (a) If A is linearly dependent, then so is B.
   (b) If B is linearly independent, then so is A.
   (c) If A spans V, then so is B.