EXAM I

Name:

Instructions: Show your work and explain every step. You may not be given credit at all for incomplete solutions. Answers with no explanations will not be accepted. Do not write the numbers in the decimal form (keep them as fractions). Cross out any unwanted solutions or writing, and circle the final answer. Do not try to mislead me. When there is choice, indicate clearly which parts you want me to grade.

(1) (15 points) Find $\det\left((A^{10})^{-1}\right)$, where

$$A = \begin{bmatrix} 5 & 0 & 0 & 0 \\ 9 & 4 & 6 & 5 \\ 8 & 0 & 3 & 7 \\ 7 & 0 & 0 & 2 \end{bmatrix}.$$

(2) Let

$$A = \begin{bmatrix} 2 & 3 & 4 \\ 3 & 2 & 2 \\ 1 & 2 & 4 \end{bmatrix}.$$

(a) (15 points) Solve the following system by Gaussian elimination (do it on back):

$$\begin{aligned} 2x_1 + 3x_2 + 4x_3 &= 3. \\ 3x_1 + 2x_2 + 2x_3 &= 2. \\ x_1 + 2x_2 + 4x_3 &= 4. \end{aligned}$$

(b) (5 points) Use the elementary row operations that you performed in the first part of this question to find $\det(A)$. If you find the determinant by another method, you will get no credit.

(c) (10 points) Find A^{-1} either by the adjoint method or by Gauss-Jordan elimination.

(d) (10 points) Now solve the following system by the inverse method.

$$2x_1 + 3x_2 + 4x_3 = 3.$$
$$3x_1 + 2x_2 + 2x_3 = 2.$$
$$x_1 + 2x_2 + 4x_3 = 4.$$

(3) (16 points) Do only two of the following 4 parts. Make it clear which ones you're doing:

(a) Let A be an $n \times n$ matrix such that $A^2 = A$ and let $B = I - A$. What is $\det(A)$? Also, prove that $B^2 = B$.

(b) Prove or disprove the following (if the statement is false, write down a counterexample; if it's true, prove it):
Let $n > 1$. If A and B are $n \times n$ matrices such that $AB = 0$, then either $A = 0$ or $B = 0$.

(c) Let A be an $n \times n$ matrix such that $A^k = I$, for some positive integer k. Prove that A is invertible. What is A^{-1}? Write the inverse in terms of powers of A.

(d) Let a be an $n \times 1$ vector such that $a \cdot a = 1$, and let $A = I - 2aa^T$. Prove that A is symmetric and orthogonal and $A^{-1} = A$.

(4) (15 points) Decide whether the following are true or false. Do only 15 parts and circle the ones you decide to do. If the statement is true, write TRUE to the left of the statement and if it's false write FALSE to the left of the statement.

(a) The inverse of a nonsingular upper-triangular (respectively lower-triangular) matrix is upper-triangular (respectively lower-triangular).

(b) If A is an upper-triangular (respectively lower-triangular) matrix and A is symmetric, then A is diagonal.

(c) The inverse of a nonsingular diagonal matrix is diagonal.

(d) The determinant of a diagonal or lower-triangular or upper-triangular matrix is equal to the product of the elements of its main diagonal.

(e) Let A be an $n \times n$ matrix. Then A is nonsingular if and only if the system $Ax = 0$ has only the trivial solution.

(f) If a homogeneous systems of equations has a non-trivial solution, then it has infinitely many solutions.

(g) Let A be a square matrix. If B is a matrix obtained from A by interchanging two rows (or two columns), then $\det(B) = -\det(A)$.

(h) The sum of two orthogonal matrices is orthogonal.

(i) Every square matrix can be written as a sum of a symmetric matrix and a skew-symmetric matrix.

(j) The trace of a skew-symmetric matrix is 0.

(k) Let A be an $n \times n$ skew-symmetric matrix and let $x \in \mathbb{R}^n$ (i.e. x is an $n \times 1$ real vector). Then, $x^T Ax = 0$.

(l) If A and B are $n \times n$ matrices, then $(AB)^T = B^T A^T$.

(m) The determinant of an orthogonal matrix is either 0 or 1.

(n) If A and B are $n \times n$ matrices, then $AB = BA$.

(o) If A and B are invertible $n \times n$ matrices, then $(AB)^{-1} = A^{-1}B^{-1}$.

(p) Let A be an $n \times n$ matrix and let r be a real number. Then, $\det(rA) = r \det(A)$.

(q) Let A be a square matrix. If B is a matrix obtained from A by adding a multiple of one row of A to another row of A, then $\det(B) = -\det(A)$.

(r) The matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}$ is an elementary matrix.

(s) Let A and B be $n \times n$ matrices. Then $\det(A^9 B^T) = ((\det(A))^9)\,(\det(B))$.

(t) If a matrix A is row equivalent to a matrix B, then B is row equivalent to A. If, in addition, A is also row equivalent to the identity matrix, then both A and B are invertible.

(u) Let A be an $n \times n$ matrix. If A is symmetric and skew-symmetric, then $A = 0$.

(5) (a) (8 points) Find $(I - 2C^T)^2$ and C^{-1}, where

$$C = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

(b) (2 points) Let $u = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$ and $v = \begin{bmatrix} 1 \\ 5 \\ 6 \end{bmatrix}$. Find $u \cdot v$.

(c) (4 points) Let $D = \operatorname{diag}(3, 4, 5, 6, 7, 8, 9)$. What is D^{-1}?