Quiz 2

Instructions: Show your work and explain **every** step. You may not be given credit at all for incomplete solutions. Answers with no explanations will not be accepted. Do not write the numbers in the decimal form (keep them as fractions). Do each question on a separate sheet of paper, write your name on each sheet, and staple them. **Time:** 25 minutes.

(1) (4 points) Decide if the following set S spans $\mathbb{R}^3$. If the given set does not span $\mathbb{R}^3$, give an example of a vector in $\mathbb{R}^3$ that does not belong to the span of the given set. Give full explanation for each step:

$$S = \left\{ v_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \; v_2 = \begin{bmatrix} 0 \\ -3 \\ 1 \end{bmatrix}, \; v_3 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \; v_4 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \right\}.$$

Solution: The reduced row-echelon form of

$$\begin{bmatrix} 1 & 0 & 2 & 1 & a \\ -1 & -3 & 1 & 2 & b \\ 1 & 1 & 1 & 0 & c \end{bmatrix}$$

is

$$\begin{bmatrix} 1 & 0 & 2 & 1 & a \\ 0 & 1 & -1 & -1 & c-a \\ 0 & 0 & 0 & 0 & 3c-2a+b \end{bmatrix}.$$

This system is inconsistent when $3c-2a+b \neq 0$. Thus, the given set does not span all vectors (a,b,c) with $3c-2a+b \neq 0$. For example, if you choose $c=a=0$ and $b=1$, then the vector $(0,1,0)$ is not in span S.

(2) (4 points) Determine if the following set S is a basis for $\mathbb{R}^3$. If the given set is a basis for $\mathbb{R}^3$, write the vector $(1,2,3)$ as a linear combination of the vectors in S. Give full explanation for each step:

$$S = \left\{ v_1 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix},\ v_2 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix},\ v_3 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \right\}.$$

Solution: Since S contains 3 vectors and $\dim \mathbb{R}^3 = 3$, then all we need to do to find out whether S is a basis for $\mathbb{R}^3$ or not, is to check if S is linearly independent. Now since the reduced row echelon form of

$$\begin{bmatrix} -1 & 1 & 1 & 0 \\ 1 & 0 & -1 & 0 \\ 1 & 2 & 0 & 0 \end{bmatrix}$$

is

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix},$$

we conclude that the only solution to the system $c_1v_1 + c_2v_2 + c_3v_3 = 0$ is 0. Hence, S is linearly independent, and therefore, it's a basis for $\mathbb{R}^3$.

To write $v = (1, 2, 3)$ as a linear combination of the vectors in S, we need to solve the system $c_1v_1 + c_2v_2 + c_3v_3 = v$. Now since the reduced row echelon form of $\begin{bmatrix} -1 & 1 & 1 & 1 \\ 1 & 0 & -1 & 2 \\ 1 & 2 & 0 & 3 \end{bmatrix}$ is $\begin{bmatrix} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -5 \end{bmatrix}$, we conclude that $c_1 = -3$, $c_2 = 3$, and $c_3 = -5$.

(3) (2 points) Determine if the following set S of polynomials form a basis for P_2. Give full explanation:

$$S = \{v_1 = -1 + t + t^2,\ v_2 = 1 + 2t^2,\ v_3 = 1 - t\}$$

Solution: Since S contains 3 vectors and $\dim P_2 = 3$, then all we need to do is to check if S is linearly independent. Thus, we need to check whether the system $c_1v_1 + c_2v_2 + c_3v_3 = 0$ has only the trivial solution. The augmented matrix corresponding to this system is $\begin{bmatrix} -1 & 1 & 1 & 0 \\ 1 & 0 & -1 & 0 \\ 1 & 2 & 0 & 0 \end{bmatrix}$ which is the same as the one in the previous example. Thus, we conclude that S is linearly independent, and hence, it's a basis for P_2.

(4) (3 points) Find a basis for $\mathbb{R}^4$ that includes the following vectors. Give full explanation: $\left\{ v_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \\ -1 \end{bmatrix} \right\}$.

Solution: We form a matrix whose first two columns are v_1 and v_2 and whose remaining 4 columns are the columns of the 4×4 identity matrix. This matrix is:

$$A = \begin{bmatrix} 1 & -1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Now we find the reduced row echelon form of the above matrix which is

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 & -1 \end{bmatrix}.$$

Now we take the columns (make them as column vectors) of A corresponding to the columns with the leading 1's in B. Since the leading 1's in B are in columns 1, 2, 3, and 4, then $\{v_1, v_2, e_1, e_2\}$ are basis for $\mathbb{R}^4$ containing the given vectors, where e_1 is the first column of the 4×4 identity matrix I_4 and e_2 is the second column of I_4.

(5) (3 points) Let

$$S = \left\{ v_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} 0 \\ -3 \\ 1 \end{bmatrix}, v_3 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, v_4 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \right\}.$$

Find a basis from S for span S. Give full explanation.

Solution: We form a matrix whose columns are the given vectors in order. This matrix is:

$$A = \begin{bmatrix} 1 & 0 & 2 & 1 \\ -1 & -3 & 1 & 2 \\ 1 & 1 & 1 & 0 \end{bmatrix}.$$

Now we find the reduced row echelon form of the above matrix which is

$$B = \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Now we take the columns (make them as column vectors) of A corresponding to the columns with the leading 1's in B. Since the leading 1's in B are in columns 1 and 2, then $\{v_1, v_2\}$ are basis from S for span S.

(6) (4 points, 2 points each) Decide if the following set W is a subspace of the given vector space V. Give full explanation:

(a) $V = \mathbb{R}^3$. W is the set of all vectors in V of the form $(5a, 2a, 2)$.

Solution: W is not a subspace because it does not contain the zero vector.

(b) $V = M_{nn}$. W is the set of all $n \times n$ symmetric matrices.

Solution: We need to check closure under addition and under scalar multiplication.

Closure under addition: Let A and B be in W. Then $(A+B)^T = A^T + B^T = A + B$. Thus, $A+B$ is symmetric which means $A+B \in W$.

Closure under scalar multiplication: Let A be in W and α be a scalar. Then $(\alpha A)^T = \alpha A^T = \alpha A$. Thus, αA is symmetric which means $\alpha A \in W$.

Therefore, W is a subspace of M_{nn}.

Matrix	Its reduced row echelon form
$\begin{bmatrix} 1 & 0 & 2 & 1 & a \\ -1 & -3 & 1 & 2 & b \\ 1 & 1 & 1 & 0 & c \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 2 & 1 & a \\ 0 & 1 & -1 & -1 & c-a \\ 0 & 0 & 0 & 0 & 3c-2a+b \end{bmatrix}$
$\begin{bmatrix} -1 & 1 & 1 & 0 \\ 1 & 0 & -1 & 0 \\ 1 & 2 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$
$\begin{bmatrix} -1 & 1 & 1 & 1 \\ 1 & 0 & -1 & 2 \\ 1 & 2 & 0 & 3 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -5 \end{bmatrix}$
$\begin{bmatrix} 1 & -1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 & -1 \end{bmatrix}$
$\begin{bmatrix} 1 & 0 & 2 & 1 \\ -1 & -3 & 1 & 2 \\ 1 & 1 & 1 & 0 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$