Review

Review the following: definition of functions, domain, target, range, image, preimage, one-to-one (injective), onto (surjective), bijection (one-to-one correspondence), composition of functions.

Review from Calculus:

Prove or disprove the following: Let $f: A \longrightarrow B$ and $g: B \longrightarrow C$ be functions, let $\phi \neq S \subseteq A$ and $\phi \neq T \subseteq B$, let G be preimage of the image of S and H the image of the preimage of T, and let a, b, c, and d be real numbers and let $a < b$ and $c < d$. (For the first three parts, if the statement is false and if one direction is true, indicate that, and give a counterexample to show the other direction is false.) Then

(1) $g \circ f$ is one-to-one iff g and f are one-to-one.
(2) $g \circ f$ is onto iff g and f are onto.
(3) $g \circ f$ is bijection iff g and f are bijection.
(4) $G = S$. If false, indicate if one of these two sets is a subset of the other.
(5) $H = T$. If false, indicate if one of these two sets is a subset of the other.
(6) Prove that the function $f : (\mathbb{N} \cup \{0\}) \times (\mathbb{N} \cup \{0\}) \longrightarrow \mathbb{N} \cup \{0\}$, defined by $f(k, n) = 2^k(2n + 1) - 1$ is one-to-one.
(7) Give an example of a bijection from (a, b) to (c, d).
(8) Give an example of a bijection from $\mathbb{R}$ to (a, ∞).
(9) Give an example of a bijection from $(0, 1)$ to (a, ∞).

Definitions: Let V and W be vector spaces and let L be a function from V into W. L is called a *linear transformation* if it satisfies the following two conditions:

(i) $L(u + v) = L(u) + L(v)$, for all u and v in V.
(i) $L(\beta u) = \beta L(u)$, for every u in V and every scalar β.

Remarks:

(1) If $W = V$, we call L a linear operator on V.

(2) The above two conditions can be replaced by:

$L(\beta u + \gamma v) = \beta L(u) + \gamma L(v)$, for all u and v in V and for all scalars β and γ.

(3) Some people write Lu instead of $L(u)$.

Definition: Let L be a linear transformation from a vector space V into a vector space W and M be a linear transformation from W into a vector space X .

(1) If $W = V$ and $L(v) = v$ for all $v \in V$, then L is called the identity operator on V and it's denoted by I.
(2) range L (the range of L; i.e. $L(V)) = \{w \in W \mid w = L(v), \text{ for some } v \in V\}$.
(3) If range $L = W$, we say L is onto.
(4) L is called one-to-one iff for all v_1 and v_2 in V, if $L(v_1) = L(v_2)$, then $v_1 = v_2$.
(5) ker L (the kernel of $L) = \{v \in V \mid L(v) = 0\}$.
(6) The dimension of ker L is called the nullity of L, denoted nullity L.
(7) The dimension of range L is called the rank of L, denoted rank L.
(8) If $S \subseteq V$, img S (the image of S; i.e. $L(S)) = \{w \in W \mid w = L(v), \text{ for some } v \in S\}$.
(9) If $T \subseteq W$, preimage T (the preimage of T) is $\{v \in V \mid L(v) \in T\}$.
(10) The product ML is the function defined as follows: $ML(u) = (M \circ L)(u) = M(L(u)), \forall u \in V$.

Theorem: Let L be a linear transformation from a vector space V into a vector space W and let M be a linear transformation from W into a vector space X. Then

(1) $L(0) = 0$.
(2) ML is a linear transformation.
(3) $L(u - v) = L(u) - L(v)$, for all u and v in V.
(4) If $S = \{v_1, v_2, \cdots, v_k\}$ spans V, then $T = L(S) = \{L(v_1), L(v_2), \cdots, L(v_k)\}$ spans range L. (In particular, if S is a basis for V and $u \in$ range L, then u is completely determined by members of T.
(5) The image of a subspace of V is a subspace of W. In particular, range L is a subspace of W.

(6) The preimage of a subspace of W is a subspace of V.

(7) If L_2 is also a linear transformation from V into W and $\{v_1, v_2, \cdots, v_n\}$ is a basis for V. If $L(v_i) = L_2(v_i)$ for $i = 1, 2, \cdots, n$, then $L(u) = L_2(u)$, for each $u \in V$.

(8) ker L is a subspace of V.

(9) L is one-to-one iff ker $L = \{0\}$.

(10) rank L + nullity L = dim V.

(11) If dim V = dim W and L is one-to-one (respectively onto), then L is onto (respectively one-to-one).

(12) If L is onto, then dim $W \leq$ dim V.

(13) L is one-to-one iff the image of every linearly independent subset of V is linearly independent in W.

(14) If dim V = dim W, then L is one-to-one iff the image of every basis for V is a basis for W.

(15) If L is a bijection, then the image of every basis for V is a basis for W.

Theorem: Let L be a function from a vector space V into a vector space W. If $L(0) \neq 0$, then L is not a linear transformation.

Remark: Read the examples in Section 10.1 especially those about projection, rotation, dilation, contraction, and reflection. These are important.

Examples of linear transformations:

(1) (Projection) $L : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$, $L(x, y, z) = (x, y)$.

(2) (Dilation) $L : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$, $L(u) = ru$, $r > 1$.

(3) (Contraction) $L : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$, $L(u) = ru$, $0 < r < 1$.

(4) (Reflection with respect to the x-axis) $L : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, $L(x, y) = (x, -y)$.

(5) (Rotation counterclockwise by θ) $L : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, $L(u) = Au$, where

$$\begin{bmatrix} \cos\,\theta & -\sin\,\theta \\ \sin\,\theta & \cos\,\theta \end{bmatrix}.$$