Homework 5: Real Vector Spaces and Supspaces

(1) Determine whether the following sets V are closed under $\oplus$ and $\odot$

(a) V is the set of all ordered pairs of real numbers (x, y), where $x > 0$ and $y > 0$; $(x, y) \oplus (u, v) = (x + u, y + v)$, $c \odot (x, y) = (cx, cy)$.

Answer: Closed under $\oplus$ but not under $\odot$. Hence, it is not a vector space.

(b) V is the set of all ordered triples of real numbers of the form $(0, x, y)$; $(0, x, y) \oplus (0, u, v) = (0, x + u, y + v)$, $c \odot (0, x, y) = (0, 0, cz)$.

Answer: Closed under both $\oplus$ and $\odot$. But, it is not a vector space because axiom (h) is not satisfied (a counterexample was given in class).

(c) V is the set of all polynomials of the form $at^2 + bt + c$, where a, b, and c, are real numbers with $b = a + 1$; $(a_1t^2 + b_1t + c_1) \oplus (a_2t^2 + b_2t + c_2) = (a_1 + a_2)t^2 + (b_1 + b_2)t + (c_1 + c_2)$, $r \odot (at^2 + bt + c) = rat^2 + rbt + rc$.

Answer: Not closed under both $\oplus$ and $\odot$. Hence, it is not a vector space.

(2) Determine whether the given set V with the given oparations is a vector space:

(a) V is the set of all ordered pairs of real numbers (x, y); $(x, y) \oplus (u, v) = (x + u, y + v)$, $c \odot (x, y) = (0, 0)$.

Answer: Axiom (h) does not hold. Hence, it is not a vector space.

(b) V is the set of all real numbers; $u \oplus v = 2u - v$, $c \odot u = cu$.

Answer: Axiom (a) does not hold. Hence, it is not a vector space.

(c) V is the set of all positive real numbers; $u \oplus v = uv$, $c \odot u = u^c$.

Answer: vector space.

(3) Show that if $u \neq 0$ and $a \odot u = b \odot u$, then $a = b$.

(4) Which of the following subsets of $\mathbb{R}^3$ are subspaces of $\mathbb{R}^3$? The set W of all vectors of the form

(a) (a, b, c), where $a = c = 0$.

Answer: subspace.

(b) (a, b, c), where $a = -c$.

Answer: subspace.

(c) (a, b, c), where $b = 2a + 1$.

Answer: Not a subspace.

(5) Which of the following subsets of $\mathbb{R}^2$ are subspaces of $\mathbb{R}^2$? The set W of all vectors of the form

(a) The union of the first and the third quadrant.

Answer: Not a subspace.

(b) The set of all points in the unit disk (on and inside the unit circle).

Answer: Not a subspace.

(6) Which of the following subsets of P_2 are subspaces of P_2? The set W of all polynomials of the form

(a) $a_0 + a_1 t + a_2 t^2$, where $a_0 = a_1 = 0$.

Answer: subspace.

(b) $a_0 + a_1 t + a_2 t^2$, where $a_1 = 2a_0$.

Answer: subspace.

(c) $a_0 + a_1 t + a_2 t^2$, where $a_0 + a_1 + a_2 = 2$.

Answer: Not a subspace.

(d) $a + t^2$.

Answer: Not a subspace.

(e) $a_0 + a_1 t + a_2 t^2$, where $a_0,\ a_1,\ a_2$ are integers.

Answer: Not a subspace.

(f) $p(t) = a_0 + a_1 t + a_2 t^2$, where $p(0) = 0$.

Answer: subspace.

(7) Which of the following subsets of M_{23} are subspaces of M_{23}? The set W of all matrices of the form

(a) $\begin{bmatrix} a & b & c \\ d & 0 & 0 \end{bmatrix}$, where $b = a + c$.

Answer: subspace.

(b) $\begin{bmatrix} a & b & c \\ d & 0 & 0 \end{bmatrix}$, where $c > 0$.

Answer: Not a subspace.

(c) $\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$, where $a = -2c$ and $f = 2e + d$.

Answer: subspace.

(8) Which of the following subsets of M_{nn} are subspaces of M_{nn}?

(a) The set of all $n \times n$ symmetric matrices.

Answer: subspace.

(b) The set of all $n \times n$ nonsingular matrices.

Answer: Not a subspace.

(c) The set of all $n \times n$ diagonal matrices.

Answer: subspace.

(d) The set of all $n \times n$ singular matrices.

Answer: Not a subspace.

(e) The set of all $n \times n$ upper triangular matrices.

Answer: subspace.

(f) The set of all $n \times n$ matrices whose determinant is 1.

Answer: Not a subspace.

(9) Which of the following subsets are subspaces of $C(-\infty, \infty)$

(a) All nonnegative functions.

Answer: Not a subspace.

(b) All constant functions.

Answer: subspace.

(c) All functions f such that $f(0) = 0$.

Answer: subspace.

(d) All functions f such that $f(0) = 5$.

Answer: Not a subspace.

(e) All differentiable functions.

Answer: subspace.

(10) Let $S = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$. Determine if $u = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ belongs to span S.

Answer: Yes.

(11) Let $S = \left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \\ 6 \end{bmatrix} \right\}$. Determine if $u = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$ belongs to span S (i.e. if it's a linear combination of the vectors of S). How many vectors are in span S?

Answer: Yes. Span S has infinitely many vectors.

Note that the third vector is S is a multiple of the second. Hence, we can remove one of these two vectors from S and still get the same span. Since one of the vectors is a multiple of the other, when you solve the system while you're trying to find out whether u is in span S, you get a singular matrix of coefficients and the corresponding system has infinitely many solutions. (But this is not the reason why span S has infinitely many vectors.)

(12) Let W be the set of all vectors of the form $\begin{bmatrix} s \\ 3s \\ 2s \end{bmatrix}$. Show that W is a subspace of $\mathbb{R}^3$ by finding a vector v in $\mathbb{R}^3$ such that $W = \text{span}\{v\}$.

Answer: $v = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$.

(13) Let W be the set of all vectors of the form $\begin{bmatrix} 5a + 2b \\ a \\ b \end{bmatrix}$. Show that W is a subspace of $\mathbb{R}^3$ by finding vectors u and v in $\mathbb{R}^3$ such that $W = \text{span}\{u, v\}$.

Answer: $u = \begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix}$, $v = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$,

(14) Let W be the set of all vectors of the form shown (a, b, and c are arbitrary real numbers). Find a set of vectors S that span W

(a) $\begin{bmatrix} 3a + b \\ 4 \\ a - 5b \end{bmatrix}$.

Answer: Not a vector space.

(b) $\begin{bmatrix} a-b \\ b-c \\ c-a \\ b \end{bmatrix}$.

Answer: $S = \{u, v, w\}$, where

$$u = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \ v = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \ w = \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix},$$