Review

(1) To check if $S = \{v_1, v_2, \cdots, v_n\}$ spans $\mathbb{R}^m$, you do the following: If $n < m$, then S does nor span $\mathbb{R}^m$. If $n \geq m$, then you form the matrix A whose columns are the vectors of S, and then you solve the system $Ac = b$ by Gaussian elimination or Gauss-Jordan elimination, where b is an arbitrary vector in $\mathbb{R}^m$. That means the elements of b should be symbols. In other words, you have to form the augmented matrix $[v_1\ v_2\ \cdots\ v_n\ b]$, and then transform it to row echelon form or reduced row echelon form and solve the new system you get. If the system is consistent (i.e. has one solution or infinitely many solutions) no matter what the vector b is, then S spans $\mathbb{R}^m$. If the system is inconsistent (i.e. has no solution) for some choices of vector b, then S does not span $\mathbb{R}^m$. In this case, the row echelon form or the reduced row echelon form of the augmented matrix must contain a row whose entries are all zeros except the last entry which could be nonzero for some values of vector b. Now if S spans $\mathbb{R}^m$ and you were asked to write one of the vectors of $\mathbb{R}^m$ (say v) as a linear combination of the vectors in S, then you need to solve the system $Ac = v$. When you do that you'll get the unkowns which are the elements of c (let's call these elements c_1, c_2, $\cdots$, c_n). After that you write $v = c_1v_1 + c_2v_2 + \cdots + c_nv_n$.

(2) To check if $S = \{v_1, v_2, \cdots, v_n\}$ is linearly independent in $\mathbb{R}^m$, you do the following: If $n > m$, then S is linearly dependent. If $n \leq m$, then you form the matrix A whose columns are the vectors of S, and then you solve the system $Ac = 0$ by Gaussian elimination or Gauss-Jordan elimination. In other words, you have to form the augmented matrix $[v_1\ v_2\ \cdots\ v_n\ 0]$, and then transform it to row echelon form or reduced row echelon form and solve the new system you get. (Note that there is no need to write the zero column at the end.) If the system has only the trivial solution, then S is linearly independent. If the system has a nontrivial solution (in this case it must have infinitely many solutions which means you get arbitary elements in the solution), then S is linearly dependent. Now if S is linearly dependent and you were asked to write one of the vectors in S as a linear combination of

the other vectors in S, then you find one of the nontrivial solutions (say it's c, where the elements of c are $c_1,\ c_2, \cdots, c_n$). Now since c is a solution, then we must have $c_1v_1 + c_2v_2 + \cdots + c_nv_n = 0$. Now solve for one of the $v_i's$ (i.e. just leave one of the v_i's on the left hand side and move everything else to the other side).

(3) To check if $S = \{v_1, v_2, \cdots, v_n\}$ is a basis for $\mathbb{R}^m$, you do the following: If $n \neq m$, then S is not a basis (if $n > m$, then S is linearly dependent and also it may not span $\mathbb{R}^m$; if $n < m$, then S does not span $\mathbb{R}^m$ and also it may be linearly dependent). If $m = n$, then it suffices to check if S is linearly independent using the method in the previous item. If S is linearly independent, then it's a basis. If S is not linearly independent, then it is not a basis.

Now if S is a basis for $\mathbb{R}^m$ and you were asked to write a given vector v in $\mathbb{R}^m$ as a linear combination of the vectors in S, then you need to solve the system $Ac = v$. When you do that you'll get the unkowns which are the elements of c (let's call these elements c_1, c_2, $\cdots$, c_n). After that you write $v = c_1v_1 + c_2v_2 + \cdots + c_nv_n$.

(4) To find a basis from $S = \{v_1, v_2, \cdots, v_n\}$ for span S, you do the following: You form the matrix $[v_1\ v_2\ \cdots\ v_n]$, and then transform it to reduced row echelon form by Gauss-Jordan elimination. Then take the vectors of S corresponding to the columns (of the reduced row echelon form) that contain the leading 1's.

(5) To extend a linearly independent set $S = \{v_1, v_2, \cdots, v_n\}$ to form a basis for $\mathbb{R}^m$ where $n < m$, you do the following: You form the matrix $[v_1\ v_2\ \cdots\ v_n\ e_1\ e_2\ \cdots\ e_m]$, and then transform it to reduced row echelon form by Gauss-Jordan elimination. Then take the vectors corresponding to the columns (of the reduced row echelon form) that contain the leading 1's. Note that e_i is the ith column of the $m \times m$ identity matrix. I.e. the $e_i's$ are the standard basis for $\mathbb{R}^m$. Note also that the basis you get here may contain vectors from S and vectors from the standard basis of $\mathbb{R}^m$.

(6) To do any of the above items for P_m, you transform the question to a question similar to the above by taking the coefficients of each polynomial you're given and writing that as a vector.