Homework 4

Remarks: Do not turn in this homework. These problems are for practice. We may go over some of them in class, but we will do that only after you try them at home. You have to show your work and explain every step. For elementary row operations, you must write down each operation you do.

(1) Find the row echelon form of the following matrix

$$\begin{bmatrix} 0 & 2 & 3 & 4 \\ 4 & 0 & 5 & 6 \\ 1 & 2 & 4 & 3 \end{bmatrix}$$

Answer: (note: we'll do a similar example in class)

$$\begin{bmatrix} 1 & 2 & 4 & 3 \\ 0 & 1 & 3/2 & 2 \\ 0 & 0 & 1 & 10 \end{bmatrix}$$

(2) Use the previous question to solve the following system

$$\begin{aligned} x_1 + 2x_2 + 4x_3 &= 3. \\ 2x_2 + 3x_3 &= 4. \\ 4x_1 + 5x_3 &= 6. \end{aligned}$$

Answer: First rearrange the equations (to take advantage of your work in the previous question) by making the second equation the first, the third the second, and the first the third. This gives you an augmented matrix equal to the matrix in the previous question. Since you have already made the matrix in the pervious question in the row echelon form, solve the corresponding system by backward substitution to get: $x_1 = -11$, $x_2 = -13$, and $x_3 = 10$.

(3) Let u and v be solutions of the homogeneous linear system $Ax = 0$.
   (a) Show that $u + v$ is a solution of the system.
   (b) If r is a number, show that ru is a solution of the system.

(4) Show that if u and v are solutions of the linear system $Ax = b$, then $u - v$ is a solution of the homogeneous system $Ax = 0$. Now assume $b \neq 0$ and assume

the system $Ax = b$ is consistent. Show that if x is a particular solution of $Ax = b$ and y is a solution to $Ax = 0$, then $x + y$ is a solution of $Ax = b$.

(5) Without computing the determinant, find all values of a for which A is invertible, where

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 2 & a \end{bmatrix}$$

Answer: Switch the first row and the second row and call the resulting matrix B. Then, B is lower-triangular and, hence, its determinant is equal to the product of the elements of its main diagonal which is $(1)(1)(a) = a$. Now, since B was obtained from A by interchanging two rows, then $\det(B) = -\det(A)$ or equivalently $\det(A) = -\det(B) = -a$. Now recall that a matrix is singular iff its determinant is zero. Hence, A is invertible iff $a \neq 0$.

(6) Show that the following matrix is nonsingular and find its inverse

$$\begin{bmatrix} \cos\,\theta & \sin\,\theta \\ -\sin\,\theta & \cos\,\theta \end{bmatrix}$$

Answer: All you need to do to show it's nonsingular is to show its determinant is not zero. But, its determinant is $\cos^2\theta + \sin^2\theta$ which is equal to 1. Now its inverse is (see how to obtain the inverse of a 2×2 matrix)

$$\frac{1}{1}\begin{bmatrix} \cos\,\theta & -\sin\,\theta \\ \sin\,\theta & \cos\,\theta \end{bmatrix} = \begin{bmatrix} \cos\,\theta & -\sin\,\theta \\ \sin\,\theta & \cos\,\theta \end{bmatrix}$$

(7) **Definition** Let B and A be $n \times n$ matrices. We say B is *similar* to A if there exists a nonsingular matrix P such that $B = P^{-1}AP$.

(a) Show that the similarity relation on matrices is an equivalence relation.

(b) Let A be an $n \times n$ matrix. We define e^A to be $\sum_{k=0}^{\infty} \frac{1}{k!}A^k$. Now let B be a matrix similar to A (i.e. $B = P^{-1}AP$). Express e^B in terms of P, and P^{-1}, and powers of A. (You should not use powers of P or powers of P^{-1}.

(8) Let A be an $n \times n$ skew-symmetric matrix and let $x \in \mathbb{R}^n$ (i.e. x is an $n \times 1$ real vector). Show that $x^TAx = 0$.

Answer: Done in class.