Eigen Structure and Diagonalization

Definition: Let A be an $n \times n$ matrix. The number λ is called an *eigenvalue* of A if there exists an $n \times 1$ nonzero vector x such that $Ax = \lambda x$. Every nonzero vector satisfying $Ax = \lambda x$ is called an *eigenvector* of A corresponding to the eigenvalue λ, and (λ, x) is called and an *eigenpair* of A. The set of all eigenvalues of A is called the *spectrum* of A, denoted $\sigma(A)$.

Remarks:

(1) Eigenvalues are also called proper values (eigen is a German word means proper) or characteristic values or latent values. Eigenvectors are also called proper vectors or characteristic vectors or latent vectors.

(2) $Ax = \lambda x$ iff $(A - \lambda I)x = 0$. Thus, λ is an eigenvalue of A iff $(A - \lambda I)x = 0$ has a nontrivial solution (i.e. the solution space is not just the zero vector) iff $A - \lambda I$ is singular iff $\det(A - \lambda I) = 0$.

(3) Eigenvectors are also called **right** eigenvectors. Left eigenvectors are defined as follows: x^T is said to be a left eigenvector of A iff $x^T A = \lambda x^T$ (i.e. x is a right eigenvector of A^T associated with the eigenvelue λ). Note that x here is a column vector. Note also that A and A^T have the same eigenvalues but not necessarily the same eigenvectors.

(4) If the eigenvalues of A are distinct, and x is a right eigenvector of A corresponding to the eigenvalue λ and y^T is a left eigenvector of A corresponding to the eigenvalue μ, where $\mu \neq \lambda$, then $y^T x = 0$; if $\mu = \lambda$, then $y^T x \neq 0$.

Definition: The polynomial $f(\lambda) = \det(A - \lambda I)$ is called the *characteristic polynomial* of A and the equation $\det(A - \lambda I) = 0$ is called the *characteristic equation* of A. The roots (zeros) of the characteristic polynomial are the eigenvalues of A.

Remarks:

(1) Some people call $g(\lambda) = \det(\lambda I - A)$ the characetreristic polynomial and $\det(\lambda I - A) = 0$ the characteristic equation. Note that the characteristic

polynomial is of degree n and $g(\lambda) = (-1)^n f(\lambda)$. Thus, $f(\lambda)$ and $g(\lambda)$ have the same roots.

(2) If you expand $f(\lambda) = \det(A - \lambda I)$, then

   (a) The coefficient of λ^n is $(-1)^n$ and the coefficient of λ^{n-1} is $(-1)^{n-1} \sum_{i=1}^n a_{ii}$.

   (b) $\det(A) = \prod_{i=1}^n \lambda_i = f(0)$, where $\lambda_1, \lambda_2, \cdots, \lambda_n$ are the eigenvalues of A (including repeated eigenvalues).

   (c) $\sum_{i=1}^n \lambda_i = \mathrm{tr}(A) = \sum_{i=1}^n a_{ii}$.

   Thus, $\lambda = 0$ is an eigenvalue of A iff A is singular iff $\det(A) = 0$ iff $f(0) = 0$.

(3) If you expand $g(\lambda)$, then you get similar things (but remember $g(\lambda) = (-1)^n f(\lambda)$). In particular, you'll have here also $\det(A) = \prod_{i=1}^n \lambda_i$ and $\mathrm{tr}(A) = \sum_{i=1}^n \lambda_i$. Thus, the determinant of A is equal to the product of its eigenvalues and the trace of A is equal to the sum of its eigenvalues.

Definition: Let λ be an eigenvalue of matrix A and let W be the set consisting of the zero vector and all vectors of A associated with λ. Then W is a subspace of $\mathbb{C}^n$ and it's called the *eigenspace* of A associated with λ. The dimension of W is called the *geometric multiplicity* of λ. The *algebraic multiplicity* of λ is the multiplicity of λ as a root of $\det(A - \lambda I) = 0$. An eigenvalue is called *simple* of its algebraic multiplicity is 1, and *multiple* if its algebraic multiplicity is greater than 1. A matrix A is called *stable* if the real part of each eigenvalue of A is negative (i.e. all the eigenvalues of A lie in the open left half plane).

Theorem: The geometric multiplicity of an eigenvalue is less than or equal to its algebraic multiplicity.

bf Examples:

(1) Let $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$. Then $\lambda = 1$ is an eigenvalue of A of algebraic multiplicity 2 and a geometric multiplicity 1. Thus, the geometric multiplicity of $\lambda = 1$ as an eigenvalue of A is *less than* its algebraic multiplicity.

(2) Let $A = \begin{bmatrix} 3 & -1 & -2 \\ 2 & 0 & -2 \\ 2 & -1 & -1 \end{bmatrix}$. Then $\lambda = 1$ is an eigenvalue of A of algebraic multiplicity 2 and a geometric multiplicity 2. Thus, the geometric multiplicity of $\lambda = 1$ as an eigenvalue of A is *equal* to its algebraic multiplicity.

Definition: An $n \times n$ matrix is called *diagonalizable* (or cane be diagonalized) iff there exists a nonsingular matrix P and a diagonal matrix D such that $P^{-1}AP = D$. If such a matrix P exists, we say P *diagonalizes* A. I.e. A is diagonalizable iff it's similar to a diagonal matrix.

Definition: If an $n \times n$ matrix has less than n linearly independent eigenvectors, A is called *defective.*

Definition: Let L be a linear operator on a vector space V. An *eigenvector* of L is a nonzero vector x in V such that $L(x) = \lambda x$ for some scalar λ. In this case, we say λ is an *eigenvalue* of L.

Remark: Let A be the matrix of a linear operator L (i.e. $L(x) = Ax$). Then the eigenvalues/eigenvectors of L and A are the same.

Note: The problem in which we have to determine eigenvalues/eigenvectors is called an eigenvalue problem.

How to find the eigenvalues and associated eigenvectors of an $n \times n$ matrix A?

(1) **Eigenvalues:** find the roots of $\det(A - \lambda I)$; i.e. solve the equation $\det(A - \lambda I) = 0$. These are the eigenvalues of A.
(2) **Eigenvectors:** For each eigenvalue λ, find a basis for the solution space of $(A - \lambda I)x = 0$. The vectors in the basis are linearly independent eigenvectors of A associated with λ.

Reminder of definitions we introduced in the past:

(1) An $n \times n$ matrix is called *nilpotent* iff $A^k = 0$ for some positive integer k.
(2) An $n \times n$ matrix is called *idempotent* iff $A^2 = A$.
(3) An $n \times n$ matrix B is said to be similar to the $n \times n$ matrix A iff there exists an invertible $n \times n$ matrix P such that $B = P^{-1}AP$. Note that similarity is an equivalence relation.

Remark: Let $x_1,\ x_2, \cdots,\ x_n$ be eigenvectors of the $n \times n$ matrix A corresponding to the eigfenvalues $\lambda_1,\ \lambda_2, \cdots, \lambda_n$, and let $T = [x_1, x_2, \cdots, x_n]$. Then $AT = [\lambda_1 x_1, \lambda_2 x_2, \cdots, \lambda_n x_n] = TD$, where $D = \text{diag}(\lambda_1, \lambda_2, \cdots, \lambda_n)$. If $x_1,\ x_2, \cdots,\ x_n$ are linearly independent, then T is nonsingular, and thus we get, $T^{-1}AT = D$, which means T diagonalizes A. Note also we can write $T^{-1}A = DT^{-1}$. Thus, the columns of T are right eigenvectors of A and the rows of T^{-1} are left eigenvectors of A (and right eigenvectors of A^T).

Theorems: Let A and B be $n \times n$ matrices.

(1) If (λ, x) is an eigenpair of A and r is a nonzero number, then $\lambda, rx)$ is an eigenpair of A, and (λ^k, x) is an eigenpair of A^k, $\forall k \in \mathbb{Z}^+$. If λ is a non-real eigenvalue and A is real, then $(\overline{\lambda}, \overline{x})$ is an eigenpair of A.
(2) A and A^T have the same eigenvalues but not necessarily the same eigenvectors.
(3) AB and BA have the same eigenvalues but not necessarily the same eigenvectors.
(4) If A is nonsingular, then (λ, x) is an eigenpair of A iff $(\frac{1}{\lambda}, x)$ is an eigenpair of A^{-1}.
(5) The eigenvalues of a diagonal/lower-triangular/upper-triangular matrix are equal to the elements of the main diagonal.
(6) A is diagonalizable iff A has n linearly independent eigenvectors. Moreover, if A is similar to a diagonal matrix D, then the elements of D are the eigenvalues of A.
(7) If A is defective, then A is non-diagonalizable.

(8) If A is nondefective and $x_1,\ x_2,\ \cdots,\ x_n$ are linearly independent eigenvectors of A corresponding to the eigenvalues $\lambda_1,\ \lambda_2,\ \cdots,\ \lambda_n$ respectively, and $P = [x_1, x_2, \cdots, x_n]$, then $P^{-1}AP = \mathrm{diag}(\lambda_1, \lambda_2, \cdots, \lambda_n)$.

(9) If A and B are similar, then they have the same eigenvalues. You can easily find a relationship between their eigenvectors.

(10) If A is idempotent and λ is an eigenvalue of A, then $\lambda = 0$ or $\lambda = 1$.

(11) If A is nilpotent and λ is an eigenvalue of A, then $\lambda = 0$.

(12) The eigenvalues of a Hermitian matrix are real and the eigenvalues of a skew-Hermitian matrix have zero real parts.

(13) If λ is an eigenvalue of a unitary matrix, then $|\lambda| = 1$.

(14) A is normal iff A has n orthogonal eigenvectors.

(15) For a 2×2 matrix, the characteristic polynomial is $\lambda^2 - \mathrm{tr}(A)\lambda + \det(A) = 0$.

(16) If A is diagonal, then so are A^T and A^k, $\forall k \in \mathbb{Z}^+$, and if A is nonsingular, then A^{-1} is also diagonalizable.

(17) If the eigenvalues of A are distinct, then A is diagonalizable.

(18) A is diagonalizable iff A has a complete set of n linearly independent eigenvectors.

Exercises:

Prove (1), (2), (4), (5), (7), (9) (here also find the relationship between the eigenvetors of A and those of B), (10), (11), (12), (13), (15), (16).