To reduce a matrix A, in which the first element of the first row is zero, to row echelon form, do the following in order: find the first (from the left) nozero column (this is called the pivotal column), find the first (from the top) nozero entry in that column (this is called the pivot), interchange the first row with the row containing the pivot, then proceed as we did before (i.e. multiply the first row by the reciprocal of the pivot, and add multiples of the first row to all other rows to make all elements below the pivot zeros). Then repeat the steps on the matrix obtained from A by deleting the first row of A.

Definition: An $m \times n$ matrix A is said to be row equivalent to an $m \times n$ matrix B if B can be obtained by applying a finite sequence of elementary row operations to A.

Theorem: Let $Ax = b$ and $Cx = d$ be two linear systems of m equations and n unkowns. If the augmented matrices $[A|b]$ and $[C|d]$ are row equivalent, then the two systems have the same solution(s).

Gaussian elimination: To solve $Ax = b$, form the augmented matrix $[A|b]$, find the row echelon form $[C|d]$ of $[A|b]$ (using elementary row operations), solve the system corresponding to $[C|d]$ by backward substitution (ignore the zero rows).

Gauss-Jordan elimination: To solve $Ax = b$, form the augmented matrix $[A|b]$, find the reduced row echelon form $[C|d]$ of $[A|b]$ (using elementary row operations), solve the system corresponding to $[C|d]$ (ignore the zero rows).

Remarks:

(1) If $Ax = b$ is inconsisstent, then the row echelon form of $[A|b]$ must contain a row in which all elements are zeros except the last one which is not zero.

(2) To find A^{-1}, where A is $n \times n$, find the reduced row echelon form of $[A|I_n]$. If A is row equivalent to I_n (which means the first n columns of the reduced row echelon form of $[A|I_n]$ form I_n), then the reduced row echelon form of $[A|I_n]$ is $[I_n|A^{-1}]$. Otherwise, A is singular.

(3) Let A be $n \times n$. Finding the reduced row echelon form of $[A|I_n]$ is equivalent to simultanously solving n systems of equations $Ax = e_i$, $i = 1, \cdots, n$, by Gauss-Jordan elimination, where e_i is the i^{th} column of I_n. In general, finding A^{-1} is equivalent to solving the n systems $Ax = e_i$, $i = 1, \cdots, n$.

(4) Let A be $n \times n$. Then, the following statements are equivalent: A is nonsingular, A is row equivalent to I_n, $Ax = 0$ has only the trivial solution, $Ax = b$ has a unique solution for every $n \times 1$ vector b.