Homework 2

Remarks: Do not turn in this homework. These problems are for practice. Some of them may be assigned as an assignment and some will be done in class. More exercises may be added later. You have to show your work and explain every step. You may not be given creit at all for incomplete solutions.

Notation: If D is an $n \times n$ diagonal matrix whose diagonal elements are $d_1,\ d_2, \cdots, d_n$, then D is sometimes written as $D = \operatorname{diag}(d_1, d_2, \cdots, d_n)$.

Definition: Let A be an $n \times n$ matrix. The *trace* of A, denoted $\operatorname{tr}(A)$, is the sum of the elements of the main diagonal of A.

(1) Let A be an $n \times n$ skew-symmetric matrix. Prove that if n is odd, then A is singular.

(2) Let $D = \operatorname{diag}(d_1, d_2, \cdots, d_n)$. Give a necessary and sufficient condition for D to be singular.

Answer: D is singular if and only if $d_i = 0$, for some $1 \leq i \leq n$.

(3) Let $D = \operatorname{diag}(d_1, d_2, \cdots, d_n)$ and assume that $d_i \neq 0$, for $i = 1, \cdots, n$. Find the inverse of D.

Done in class.

(4) (True/False) If A is an $n \times n$ skew-symmetric matrix, then $\operatorname{tr}(A) = 0$.

True. Note that the main diagonal of a skew-symmetric matrix is zero. Hence, the sum of the elements (entries) of the main diagonal (which is what we call the trace) is zero.

(5) Let x be an $n \times 1$ vector such that $x^T x = 2$. Find $(I_n - xx^T)^{-1}$. *Hint:* See P. 14 of Homework 1.

Similar to an example we did in class.

(6) Let x be an $n \times 1$ vector. Is $(I_n - xx^T)$ invertible?

(7) Find $\det(A)$, adj A, A^{-1}, the minor of a_{12} and A_{12} (the cofactor of a_{12}), where

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 0 & 1 \\ -2 & 3 & -1 \end{bmatrix}.$$

Do not write the numbers in the decimal form (keep them as fractions).

Answer: $\det(A) = 3$,

$$\text{adj } A = \begin{bmatrix} -3 & 5 & 2 \\ 0 & 1 & 1 \\ 6 & -7 & -4 \end{bmatrix}.$$

$$A^{-1} = \frac{1}{\det(A)}\text{adj } A = \begin{bmatrix} -1 & 5/3 & 2/3 \\ 0 & 1/3 & 1/3 \\ 2 & -7/3 & -4/3 \end{bmatrix}.$$

The last two items are 0, 0.

(8) Solve the following linear system by using the inverse method (i.e. by using A^{-1} where A is the matrix of coefficients):

$$x_1 + 2x_2 + x_3 = 2.$$
$$2x_2 + x_3 = -3.$$
$$-2x_1 + 3x_2 - x_3 = 4.$$

Answer: The solution is $A^{-1}b$, where $b = \begin{bmatrix} 2 \\ -3 \\ 4 \end{bmatrix}$. Thus, the solution is $\begin{bmatrix} 5 \\ 11/5 \\ -37/5 \end{bmatrix}$.

(9) Solve the following system by Gramer's Rule:

$$x_1 + 2x_2 = 2.$$
$$-2x_1 + 3x_2 = 4.$$

Answer: $x_1 = \det(A_1)/\det(A)$, $x_2 = \det(A_2)/\det(A)$, where $A = \begin{bmatrix} 1 & 2 \\ -2 & 3 \end{bmatrix}$, $A_1 = \begin{bmatrix} 2 & 2 \\ 4 & 3 \end{bmatrix}$, $A_2 = \begin{bmatrix} 1 & 2 \\ -2 & 4 \end{bmatrix}$. Thus, $x_1 = -2/7$ and $x_2 = 8/7$.