Answers for Homework 1

Remarks: Do not turn in this homework. These problems are for practice. Some of them may be assigned as an assignment and some will be done in class. More exercises may be added later.

(1) Prove that every square matrix can be written as a sum of a symmetric matrix and a skew-symmetric matrix.
Done in Class.

(2) Let A be an $n \times n$ matrix such that $A^k = 0$, for some positive integer k. Prove that A is singular.
Done in Class.

(3) Let A be an $n \times n$ matrix such that $A^2 = A$.
(a) What is $\det(A)$?
(b) If A is nonsingular, then what is A?
(c) Prove that $A^k = A$, for all $k \in \mathbb{Z}^+$.
(d) Let $B = A^T$. Prove that $B^2 = B$.
(e) Let $B = I - A$. Prove that $B^2 = B$.
Done in Class.

(4) Let A be an $n \times n$ matrix such that $A^k = I$, for some positive integer k. Prove that A is invertible. What is A^{-1}? Write the inverse in terms of powers of A.
Done in Class.

(5) Let A and B be $n \times n$ matrices and suppose there exists an invertible matrix P such that $P^{-1}AP = B$. Prive that $\det(A) = \det(B)$.
Proof: $\det(P^{-1}AP) = \det(P^{-1})\det(A)\det(P) = \frac{1}{\det(P)}\det(A)\det(P) = \det(A)$. Now, since $\det(P^{-1}AP) = \det(B)$, the result follows.

(6) Let $A = (a_{ij})$ be an $n \times n$ matrix such that $a_{i,n-i+1} = 1$, for $i = 1, \cdots, n$, and $a_{ij} = 0$ otherwise. Find $\det(A)$.
Note that $A^2 = I$. Thus, $\det(A^2) = \det(I) = 1$. Therefore, $(\det(A))^2 = 1$. The result follows.

(7) (True or False) If A is a real $n \times n$ matrix, then $\det(A^TA) \geq 0$.
True. Here is why. $\det(A^TA) = \det(A^T)\det(A) = \det(A)\det(A) = (\det(A))^2$. Now the result follows from the fact that $\det(A)$ is a real number and the square of any real number is non-negative. (Note that the graph of $f(x) = x^2$ lies above the $x-$axis except at one point which is the origin; there the graph touches the x-axis without crossing to the negative part.)

(8) Prove or disprove: If A is an $n \times n$ symmetric matrix, then A^TA and AA^T are symmetric.
Done in Class.

(9) (True or False) If A is an $n \times n$ symmetric lower/upper triangular matrix, then A is diagonal.
Hint was given in class.

(10) (True or False) If A is an $n \times n$ skew-symmetric lower/upper triangular matrix, then $A = 0$.

(11) Prove or disprove: Let A and B be two matrices for which AB is defined. If $AB = 0$, then $A = 0$ or $B = 0$.
Done in Class.

(12) Prove or disprove: Let u and v be two $n \times 1$ vectors. If $u \cdot v = 0$, then $u = 0$ or $v = 0$.
Done in Class.

(13) Let A be an $n \times n$ orthogonal matrix. Prove that $\det(A) = \pm 1$.
Since A is orthogonal, then $A^{-1} = A^T$. Now since $AA^{-1} = I$, then $AA^T = I$. Thus, $\det(A)\det(A^T) = \det(I) = 1$. But, $\det(A^T) = \det(A)$. Thus, $(\det(A))^2 = 1$. Now the result follows (note that $\det(A)$ is a real number and the solution (if x is real) of the equation $x^2 = 1$ is $x = \pm 1$).

(14) Let a be an $n \times 1$ vector such that $a \cdot a = 1$, and let $A = I - 2aa^T$. Prove that A is symmetric and orthogonal and $A^{-1} = A$.
Done in Class.

(15) Let A and B be $n \times n$ matrices such that $3A - AB = -I$. What is A^{-1}?
Note that $A(3I - B) = -I$. Thus, $A(B - 3I) = I$. Hence, $A^{-1} = B - 3I$.

(16) Let A and B be $n \times n$ symmetric matrices and let r be a nonzero number. Prove or disprove the following:
   (a) rA is symmetric.
   (b) $-A$ is symmetric.
   (c) AB is symmetric.
   (d) $A + B$ is symmetric.
   (e) A^T is symmetric.
   (f) If A is invertible, then A^{-1} is symmetric.
Done in Class.

(17) Let A and B be $n \times n$ skew-symmetric matrices and let r be a nonzero number. Prove or disprove the following:
   (a) rA is skew-symmetric.
   (b) $-A$ is skew-symmetric.
   (c) AB is skew-symmetric.
   (d) $A + B$ is skew-symmetric.
   (e) A^T is skew-symmetric.
   (f) If A is invertible, then A^{-1} is skew-symmetric.
Similar to the above.

(18) Let A and B be $n \times n$ orthogonal matrices and let r be a nonzero number. Prove or disprove the following:
(a) rA is orthogonal.
(b) $-A$ is orthogonal.
(c) AB is orthogonal.
(d) $A+B$ is orthogonal.
(e) A^T is orthogonal.
(f) A^{-1} is orthogonal.
Done in Class.

(19) Find $\det(A)$ and $\det(B^9)$, where

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 0 & 2 & 3 \\ 1 & 0 & 2 & 4 \\ 3 & 0 & 1 & 2 \end{bmatrix},$$

$$B = \begin{bmatrix} 2 & 5 & 6 & 8 \\ 0 & 3 & 9 & -8 \\ 0 & 0 & 2 & 17 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Done in Class.

(20) Let A be a matrix whose inverse is

$$A^{-1} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ -2 & 3 & -1 \end{bmatrix}$$

Find $(A^2)^{-1}$ and A^{-2}.
$(A^2)^{-1} = A^{-2} = (A^{-1})^2$. Now note that

$$(A^{-1})^2 = \begin{bmatrix} 3 & 21 & 12 \\ 12 & 51 & 36 \\ 12 & 8 & 13 \end{bmatrix}$$

(21) Let

$$a = \begin{bmatrix} 2 \\ 3 \\ -4 \\ 5 \end{bmatrix}, b = \begin{bmatrix} -1 \\ -5 \\ 4 \\ 2 \end{bmatrix}, A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{bmatrix}, B = \begin{bmatrix} 2 & 1 & 3 \\ -1 & 2 & 0 \\ -3 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}.$$

Find the following if they are defined
(a) $a \cdot b$.
Answer: -23.

(b) $a - 2b$.

Answer: $\begin{bmatrix} 4 \\ 13 \\ -12 \\ 1 \end{bmatrix}$.

(c) Ab.

Answer: $\begin{bmatrix} 9 \\ 9 \end{bmatrix}$.

(d) AB.

Answer:

$$\begin{bmatrix} -9 & 17 & 42 \\ -17 & 45 & 98 \end{bmatrix}.$$

(e) BA.

Undefined.

(f) A^T.

Answer:

$$\begin{bmatrix} 1 & 5 \\ 2 & 6 \\ 3 & 7 \\ 4 & 8 \end{bmatrix}.$$

(g) $3I - A$.

Undefined. Note that I must be a square matrix.

(22) Let

$$C = \begin{bmatrix} 1 & 2 & -2 \\ 2 & 0 & 3 \\ -2 & 3 & -7 \end{bmatrix}, D = \begin{bmatrix} 2 & 3 & -2 \\ -3 & 0 & 5 \\ 2 & -5 & 0 \end{bmatrix}.$$

(a) Is C symmetric? Is it skew-symmetric? Is it orthogonal?

C is symmetric, but not skew-symmetric and not orthogonal. In order for C to be orthogonal, we must have $C^{-1} = C^T$, which implies $CC^T = I$. Now find CC^T to confirm it is not equal to I. (Note that $CC^T = C^2$ here because C is symmetric.)

(b) Is D symmetric? Is it skew-symmetric? Is it orthogonal?

Answer: D is not symmetric and it is not skew-symmetric (note that the main diagonal of a skew-symmetric matrix must be zero), and D is not orthogonal (see the above part to know how to prove that).

(c) Find $\det(C^{19})$, $\det\left((C^{19})^{-1}\right)$, $\det\left((C^T)^{19}\right)$, $\det(3C)$.

Answers: Note first $\det(C) = -5$. Thus, $\det(C^{19}) = (\det(C))^{19} = (-5)^{19}$. The remaining answers are $(-5)^{-19}$, $(-5)^{19}$, -15.