Homework 1

Remarks: Do not turn in this homework. These problems are for practice. Some of them may be assigned as an assignment and some will be done in class. More exercises may be added later.

(1) Prove that every square matrix can be written as a sum of a symmetric matrix and a skew-symmetric matrix.
(2) Let A be an $n \times n$ matrix such that $A^k = 0$, for some positive integer k. Prove that A is singular.
(3) Let A be an $n \times n$ matrix such that $A^2 = A$.
   (a) What is $\det(A)$?
   (b) If A is nonsingular, then what is A?
   (c) Prove that $A^k = A$, for all $k \in \mathbb{Z}^+$.
   (d) Let $B = A^T$. Prove that $B^2 = B$.
   (e) Let $B = I - A$. Prove that $B^2 = B$.
(4) Let A be an $n \times n$ matrix such that $A^k = I$, for some positive integer k. Prove that A is invertible. What is A^{-1}? Write the inverse in terms of powers of A.
(5) Let A and B be $n \times n$ matrices and suppose there exists an invertible matrix P such that $P^{-1}AP = B$. Prive that $\det(A) = \det(B)$.
(6) Let $A = (a_{ij})$ be an $n \times n$ matrix such that $a_{i,n-i+1} = 1$, for $i = 1, \cdots, n$, and $a_{ij} = 0$ otherwise. Find $\det(A)$.
(7) (True or False) If A is a real $n \times n$ matrix, then $\det(A^T A) \geq 0$.
(8) Prove or disprove: If A is an $n \times n$ symmetric matrix, then $A^T A$ and AA^T are symmetric.
(9) (True or False) If A is an $n \times n$ symmetric lower/upper triangular matrix, then A is diagonal.
(10) (True or False) If A is an $n \times n$ skew-symmetric lower/upper triangular matrix, then $A = 0$.
(11) Prove or disprove: Let A and B be two matrices for which AB is defined. If $AB = 0$, then $A = 0$ or $B = 0$.
(12) Prove or disprove: Let u and v be two $n \times 1$ vectors. If $u \cdot v = 0$, then $u = 0$ or $v = 0$.
(13) Let A be an $n \times n$ orthogonal matrix. Prove that $\det(A) = \pm 1$.
(14) Let a be an $n \times 1$ vector such that $a \cdot a = 1$, and let $A = I - 2aa^T$. Prove that A is symmetric and orthogonal and $A^{-1} = A$.
(15) Let A and B be $n \times n$ matrices such that $3A - AB = -I$. What is A^{-1}?
(16) Let A and B be $n \times n$ symmetric matrices and let r be a nonzero number. Prove or disprove the following:
   (a) rA is symmetric.

(b) $-A$ is symmetric.
(c) AB is symmetric.
(d) $A + B$ is symmetric.
(e) A^T is symmetric.
(f) If A is invertible, then A^{-1} is symmetric.

(17) Let A and B be $n \times n$ skew-symmetric matrices and let r be a nonzero number. Prove or disprove the following:
(a) rA is skew-symmetric.
(b) $-A$ is skew-symmetric.
(c) AB is skew-symmetric.
(d) $A + B$ is skew-symmetric.
(e) A^T is skew-symmetric.
(f) If A is invertible, then A^{-1} is skew-symmetric.

(18) Let A and B be $n \times n$ orthogonal matrices and let r be a nonzero number. Prove or disprove the following:
(a) rA is orthogonal.
(b) $-A$ is orthogonal.
(c) AB is orthogonal.
(d) $A + B$ is orthogonal.
(e) A^T is orthogonal.
(f) A^{-1} is orthogonal.

(19) Find $\det(A)$ and $\det(B^9)$, where

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 0 & 2 & 3 \\ 1 & 0 & 2 & 4 \\ 3 & 0 & 1 & 2 \end{bmatrix},$$

$$B = \begin{bmatrix} 2 & 5 & 6 & 8 \\ 0 & 3 & 9 & -8 \\ 0 & 0 & 2 & 17 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(20) Let A be a matrix whose inverse is

$$A^{-1} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ -2 & 3 & -1 \end{bmatrix}$$

Find $(A^2)^{-1}$ and A^{-2}.

(21) Let

$$a = \begin{bmatrix} 2 \\ 3 \\ -4 \\ 5 \end{bmatrix}, b = \begin{bmatrix} -1 \\ -5 \\ 4 \\ 2 \end{bmatrix}, A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{bmatrix}, B = \begin{bmatrix} 2 & 1 & 3 \\ -1 & 2 & 0 \\ -3 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}.$$

Find the following if they are defined

(a) $a \cdot b$.
(b) $a - 2b$.
(c) Ab.
(d) AB.
(e) BA.
(f) A^T.
(g) $3I - A$.

(22) Let

$$C = \begin{bmatrix} 1 & 2 & -2 \\ 2 & 0 & 3 \\ -2 & 3 & -7 \end{bmatrix}, D = \begin{bmatrix} 2 & 3 & -2 \\ -3 & 0 & 5 \\ 2 & -5 & 0 \end{bmatrix}.$$

(a) Is C symmetric? Is it skew-symmetric? Is it orthogonal?
(b) Is D symmetric? Is it skew-symmetric? Is it orthogonal?
(c) Find $\det(C^{19})$, $\det\left((C^{19})^{-1}\right)$, $\det\left((C^T)^{19}\right)$, $\det(3C)$.