Solution of FINAL EXAM

Instructions:

(1) Show your work and explain **every** step.
(2) You may not be given credit at all for incomplete solutions.
(3) **Answers with no explanations will not be accepted**.
(4) Do not write the numbers in the decimal form (keep them as fractions).
(5) If you use a method different than the method specified in the question, you will get no points.
(6) **Do NOT use calculators.**
(7) Do NOT talk to your neighbor or look at his/her paper.
(8) Use only notation used in class.
(9) **Time:** 2 hours.

Matrix	Its reduced row echelon form
$\begin{bmatrix} 1 & 1 & 1 & -1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & -2 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$
$\begin{bmatrix} 1 & 1 & -1 & 2 \\ 2 & 2 & -3 & 1 \\ -1 & -1 & 0 & -5 \end{bmatrix}$	$\begin{bmatrix} 1 & 1 & 0 & 5 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$
$\begin{bmatrix} 1 & 1 & -1 & 2 & 0 \\ 2 & 2 & -3 & 1 & 0 \\ -1 & -1 & 0 & -5 & 0 \end{bmatrix}$	$\begin{bmatrix} 1 & 1 & 0 & 5 & 0 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

(1) (8 points)

(a) Give a counterexample to prove the following statement is false:
If an $n \times n$ matrix A is row equivalent to I_n, then A and I_n have the same eigenvalues.
Solution: Let $A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$. Then A is row equivalent to I_2 (divide the first row by 2 and then the second by 2 and you'll get I_2). But, the eigenvalues of A are 2 with algebraic multiplicity 2 and the eigenvalues of I_2 are 1 with algebraic multiplicicity 2. By the way, this one was done in class.

(b) Prove that every skew-Hermitian matrix is normal.
Solution: Let A be skew-Hermitian, then $A^H = -A$. Thus, $AA^H = A(-A) = -A^2$, and $A^H A = (-A)A = -A^2$. Therefore, $AA^H = A^H A$, which means it's normal. By the way, this one was done in class.

(c) Let S be an $n \times n$ skew-symmetric matrix and n is odd, prove that $\det(S) = 0$.
Solution: We know $S^T = -S$. Thus, $\det(S^T) = \det(-S) = (-1)^n \det(S) = -\det(S)$. But, $\det(S^T) = \det(S)$. Therefore, $\det(S) = -\det(S)$. Thus, $2\det(S) = 0$, which implies $\det(S) = 0$. By the way, this one was done in class.

(d) Let S be an $n \times n$ real skew-symmetric matrix, and let (λ, x) be an eigenpair of S. Prove that if x is real, then S is singular.
Solution: $Sx = \lambda x$. Now recall that we proved that the eigenvalues of S are of the form bi, where b is a real number. But, since S is real and x is real, then Sx is real. On the other hand, $\lambda x = bix$, which is imaginary. This forces b to be zero, which means $\lambda = 0$, which implies S is singular (we said a matrix is singular iff zero is an eigenvalue of the matrix).

(2) (4 points) Let A be a 4×4 matrix with determinant of 30 and with eigenvalues of $\lambda_1 = 2$, $\lambda_2 = 3$, $\lambda_3 = 1$, and λ_4.

(a) What is the value of λ_4?

Solution: We said in class that the determinant of A is equal to the product of its eigenvalues. Thus, $30 = (2)(3)(1)\lambda_4$. Therefore, $\lambda_4 = 5$.

(b) Find $\mathrm{tr}(A)$.

Solution: We said in class that the trace of a matrix is equal to the sum of its eigenvalues. Therefore, $\mathrm{tr}(A) = 2 + 3 + 1 + 5 = 11$.

(3) (4 points) Let B be an $n \times n$ matrix similar to the $n \times n$ matrix A. I.e. there exists an invertible $n \times n$ matrix P such that $P^{-1}AP = B$. Let (λ_1, x_1), (λ_2, x_2), $\cdots$, (λ_n, x_n) be eigenpairs of A, where λ_k, $k = 1, \cdots, n$, are all the eigenvalues of A. Find corresponding eigenpairs of B (write them in terms of the eigenvalues of A and the eigenvectors of A).

Solution: $A = PBP^{-1}$. Therefore, $Ax_k = \lambda_k x_k$ iff $PBP^{-1}x_k = \lambda_k x_k$ iff $BP^{-1}x_k = \lambda_k P^{-1}x_k$ iff $(\lambda_k, P^{-1}x_k)$ is an eigenpair of B. Thus, the eigenpairs of B are $(\lambda_1, P^{-1}x_1)$, $(\lambda_2, P^{-1}x_2)$, $\cdots$, $(\lambda_n, P^{-1}x_n)$. By the way this was done in class.

(4) (6 points) Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$. Find A^{-1} by using Gauss-Jordan elimination, and also find $\det(A)$ from the Gauss-Jordan elimination steps you did.

Solution: $A^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 1 \\ 1 & 0 & -1 \end{bmatrix}$. The elementary row operations you have to do to get this are: $r_3 - r_1 \longrightarrow r_3$, $r_1 - r_2 \longrightarrow r_1$, $-r_3 \longrightarrow r_3$, $r_2 - r_3 \longrightarrow r_2$. Thus, $(-1)\det(A) = 1$. Therefore, $\det(A) = -1$.

(5) (6 points) Let $A = \begin{bmatrix} 3 & 5 & 0 \\ 2 & 0 & 8 \\ 4 & 7 & 6 \end{bmatrix}$. Find $\det(A)$ by using cofactors. Also, find $\det(A^9 A^T)^{-1}$.

Solution: $\det(A) = 3(0 - (8)(7)) - 5((2)(6) - (4)(8)) = -68$.

$\det(A^9 A^T)^{-1} = ((\det(A))^9 \cdot \det(A))^{-1} = (\det(A))^{-10} = (-68)^{-10}$.

(6) (3 points) Determine if the following set S is a basis for $\mathbb{R}^3$. Explain.

$$S = \left\{ v_1 = \begin{bmatrix} 3 \\ 2 \\ 4 \end{bmatrix}, \ v_2 = \begin{bmatrix} 5 \\ 0 \\ 7 \end{bmatrix}, \ v_3 = \begin{bmatrix} 0 \\ 8 \\ 6 \end{bmatrix} \right\}.$$

Hint: see the previous question.

Solution: When you put these vectors in a matrix, you get the matrix A in the previous question whose determinant is not zero, which means the given vectors are linearly independent. Since we have 3 vectors which is the same as the dimension of $\mathbb{R}^3$, we conclude that they form a basis for $\mathbb{R}^3$.

(7) (6 points) Decide if the following set S span $\mathbb{R}^3$. If the given set does not span $\mathbb{R}^3$, give an example of a vector in $\mathbb{R}^3$ that does not belong to the span of the given set. Explain.

$$S = \left\{ v_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, v_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, v_4 = \begin{bmatrix} -1 \\ 0 \\ -2 \end{bmatrix} \right\}.$$

Solution: From Question 4, we know the matrix that consists of the first three vectors is nonsingular. Therefore the first three vectors are linearly independent, and hence, they form a basis for $\mathbb{R}^3$, which means they span $\mathbb{R}^3$.

(8) (4 points) Let

$$S = \left\{ v_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix},\ v_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix},\ v_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix},\ v_4 = \begin{bmatrix} -1 \\ 0 \\ -2 \end{bmatrix} \right\}.$$

Find a basis from S for span S. Explain.

Solution: From the previous question, the first three vectors form a basis for Span S.

(9) (5 points) Let $v_1 = (7, 3)$ and $v_2 = (5, 2)$. Find the transition matrix for the change of basis from $[e_1, e_2]$ to $[v_1, v_2]$. Also, find the coordinates of $v = (-4, 6)$ with respect to $[v_1, v_2]$.

Solution: $A = \begin{bmatrix} 7 & 5 \\ 3 & 2 \end{bmatrix}$ is the transition matrix from $[v_1, v_2]$ to $[e_1, e_2]$. Therefore, $A^{-1} = \begin{bmatrix} -2 & 5 \\ 3 & -7 \end{bmatrix}$ is the required transition matrix. The coordinates of v are $A^{-1}v = (38, -54)$.

(10) (8 points) Let $A = \begin{bmatrix} 1 & 1 & -1 & 2 \\ 2 & 2 & -3 & 1 \\ -1 & -1 & 0 & -5 \end{bmatrix}$. Find a basis for the nullspace of A, a basis for the row space of A that does not include rows from A, and the rank of A. Indicate which one is which.

Solution: The first two rows of the reduced row echelon form of the matrix form a basis for the row space of A. Hence, the rank of A is 2. Now to find a basis for the nullspace, solve the system corresponding to the reduced row echelon form to get: $x_1 + x_2 + 5x_4 = 0$ and $x_3 + 3x_4 = 0$. Therefore, x_2 and x_4 are arbitrary and $x_3 = -3x_4$, $x_1 = -x_2 - 5x_4$. Therefore, $\{(-1, 1, 0, 0), (-5, 0, -3, 1)\}$ is a basis for the nullspace.

(11) (4 points) Determine if each of the following sets W is a subspace of the given vector space V. Explain.

(a) $V = M_{nn}$. W is the set of all $n \times n$ skew-symmetric matrices.

Solution: Let A and B be in W. Then $(A+B)^T = A^T + B^T = -A - B = -(A+B)$. Thus, $(A+B) \in W$. Now, let α be a scalar and A be in W. Then, $(\alpha A)^T = \alpha A^T = -\alpha A$. Thus, αA is in W. Hence, W is a subspace. By the way, this was done in class.

(b) $V = P_2$. W is the set of all polynomials of the form $a_0 + a_1 t + a_2 t^2$, where a_0 is an integer.

Solution: Not a subspace, because for example if you take $p(t) = 3$, then $p(t) \in W$. Now take the scalar to be $\alpha = \frac{1}{2}$. Then $\alpha p(t) \notin W$. So, not closed under scalar multiplication.

(12) (5 points) Determine if the following are linear transformations. Explain.

(a) $L: P_2 \longrightarrow P_2$, $L(at^2 + bt + c) = (2 - a)t^2 + (b - c)t$.

Solution: $L(0) = 2t^2 \neq 0$. So, not.

(b) $L: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, $L(x, y) = (x - y, y - x)$.

Solution: It be easily verified that L satisfies the two properties. Hence, it's linear. Alternatively, you can say $L(x, y) = A(x, y)$, where $A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$. We stated in class that any function of this form is a linear transformation.

(13) (6 points) Let $A = \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix}$. Find the eigenvalues of A, their geometric multiplicities (*do NOT give me the algebraic multiplicities*), and corresponding eigenvectors.

Solution: The characteristic equation is: $\lambda^2 - 4\lambda + 4 = 0$. Thus, the eigenvalues are $2, 2$. Now solve the system

$$A = \begin{bmatrix} 3-2 & -1 \\ 1 & 1-2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0,$$

to get $x_1 - x_2 = 0$, which means $x_1 = x_2$. Thus, $(1, 1)$ is the only eigenvector, which implies the geometric multiplicity is 1.

(14) (5 points) Let $f(\lambda) = (1-\lambda)(\lambda^2 + 10\lambda + 25)$. Find the eigenvalues of A and their algebraic multiplicities. (*Do NOT give me the geometric multiplicities.*) Also, find $\rho(A)$.

Solution: The eigenvalues are the roots of the characterestic polynomial which are $1, -5, -5$. Thus, the algebraic multiplicity of 1 is 1 and of -5 is 2. Now $\rho(A)$ is the maxmimum magnitude of the eigenvalues. Since we have only real eigenvalues here, then the magnitude is just the absolute value. Thus, $\rho(A) = 5$.

(15) (3 points) Let B be a 4×4 matrix. If the eigenvalues of B are the following:

- -4 with algebraic multiplicity 1 and a geometric multiplicity 1.
- -7 with algebraic multiplicity 1 and a geometric multiplicity 1.
- -9 with algebraic multiplicity 2 and a geometric multiplicity 1.

Is B diagonalizable? Explain.

Solution: We said in class that A is diagonalizable iff it has n linearly independent eigenvectors. In this case, $n = 4$. Now, the number of linearly independent eigenvectors is the sum of the geometric multiplicities which is $1 + 1 + 1 = 3$. Thus, we have only 3 linearly independent eigenvetors (less than 4). So, not.

(16) (4 points) Let $A = \begin{bmatrix} 3 & -1 \\ 2 & -5 \end{bmatrix}$. Find $\|A\|_1$ and $\|A\|_\infty$. Make it clear which one is which.

Solution: Take the absolute value of each element of A. Then find the sum of each row to get the sums are 4 and 7. Take the maximum which is 7. This is $\|A\|_\infty$. Now find the sum of each column (after taking the absolute value of each element) to get the sums are 5 and 6. Take the maximum of these which is 6. This is $\|A\|_1$.

(17) (a) (6 points) Transform the following set S to an orthonormal set using Gram-Schmidt process

$$S = \left\{ v_1 = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix},\ v_2 = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} \right\}.$$

(b) Find a nonzero vector in $\mathbb{R}^3$ that is orthogonal to v_1 and v_2, where v_1 and v_2 are as above.

(18) (8 points)

(a) Let $L:\ \mathbb{R}^3 \longrightarrow \mathbb{R}^2$, $L(x_1, x_2, x_3) = (2x_1 - x_3, x_2 - x_3 + 4x_1)$. Find the matrix of L.

Solution: $A = \begin{bmatrix} 2 & 0 & -1 \\ 4 & 1 & -1 \end{bmatrix}$.

(b) Let $L:\ \mathbb{R}^3 \longrightarrow \mathbb{R}^3$, $L(x_1, x_2, x_3) = (0, 2x_1 - x_2, 0)$. Find a basis for ker L.

Solution: Solve $L(x) = 0$ to get $2x_1 = x_2$. Thus, x_3 is arbitrary. Therefore, the kernel consists of all vectors of the form $(x_1, 2x_1, x_3)$. Hence, $\{(1, 2, 0), (0, 0, 1)\}$ is a basis for ker L.

(c) Let $L:\ P_2 \longrightarrow P_3$, $L(at^2 + bt + c) = (a - c)t^3 + b$. Find a basis for range L.

Solution: From the formula, $\{t^3, 1\}$ is a basis.

(d) Let $L:\ \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, be the linear transformation such that $L(v)$ is the rotation of v clockwise (*be careful here; it's clockwise not counterclockwise*) by θ. Find the matrix of L.

Solution: This was done in class. The matrix is $A = \begin{bmatrix} \cos\ \theta & \sin\ \theta \\ -\sin\ \theta & \cos\ \theta \end{bmatrix}$

(19) (5 points) Determine if the followng are true or false. If the item is true, write "TRUE" to the left of the item; if it's false, write "FALSE" to the left of the item:

(a) If a matrix P diagonalizes a matrix A, then P is unique. Solution: False. This was done in class.

(b) Every unitary matrix is normal.
Solution: True. This was done in class.

(c) If L is a function from a vectorspace V into a vectorspace W and $L(0) = 0$, then L is a linear transformation.
Solution: False. This was mentioned in class.

(d) If $\lambda = 0$ is an eigenvalue of an $n \times n$ matrix A, then A is singular.
Solution: True. This was mentioned in class.

(e) Let A be an $n \times n$ matrix and let $f(\lambda) = \det(A - \lambda I)$ and $g(\lambda) = \det(\lambda I - A)$, then $g(\lambda) = -f(\lambda)$.
Solution: False. This was done in class. Note that $g(\lambda) = (-1)^n f(\lambda)$.

(f) If A is an $n \times n$ matrix and B is the matrix obtained from A by adding a multiple of one row of A to another, then $\det(B) = \det(A)$.
Solution: TRue. This was mentioned in class.

(g) If V is a vector space of dimension n, then it's possible to have $n+1$ orthogonal vectors in V.
Solution: False. This was mentioned in class.

(h) The algebraic multiplicity of an eigenvalue is less than or equal to its geometric multiplicity.
Solution: False. This was mentioned in class. The opposite is what's true.

(i) The main diagonal of a Hermitian matrix is real.
Solution: True. This was mentioned in class.

(j) If a homogeneous system of linear equations has a nontrivial solution, then it has infinitely many solutions.
Solution: True. This was mentioned in class.