Assignment 4

Due: Friday, Dec 3, 04, at 2:00 pm in class

(1) Deteremine whether the following are linear transformations or not:

(a) $L: \ P_2 \longrightarrow P_1$, $L(at^2 + bt + c) = (3 - b)t + (a - c)$.

Solution: $L(0) = 3t \neq 0$. Since, $L(0) \neq 0$, then it is not a linear transformation.

(b) $L: \ \mathbb{R}^2 \longrightarrow \mathbb{R}^3$, $L(x_1, x_2) = (x_1 - x_2 + 1, 0, -x_1)$.

Solution: $L(0, 0) = (1, 0, 0)$. Since $L(0) \neq 0$, then it is not a linear transformation.

(c) $L: \ \mathbb{R}^3 \longrightarrow \mathbb{R}^2$, $L(x_1, x_2, x_3) = (x_1 - x_3, 2x_2)$.

Solution: Note that $L(0, 0, 0) = (0, 0)$, but that does not necessarily imply L is a linear transformation. We still have to check if it satisfies the two conditions we mentioned in class:

Let $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$. We have to check if $L(x + y) = L(x) + L(y)$:

$L(x+y) = L(x_1+y_1, x_2+y_2, x_3+y_3) = ((x_1 + y_1) - (x_3 + y_3), 2(x_2 + y_2)) = (x_1 + y_1 - x_3 - y_3, 2x_2 + 2y_2)$.

$L(x)+L(y) = (x_1-x_3, 2x_2)+(y_1-y_3, 2y_2) = (x_1-x_3+y_1-y_3, 2x_2+2y_2)$.

Thus, $L(x + y) = L(x) + L(y)$. But, this does not imply L is a linear transformation. We still have to check if $L(\alpha x) = \alpha L(x)$:

$L(\alpha x) = L(\alpha x_1, \alpha x_2, \alpha x_3) = (\alpha x_1 - \alpha x_3, 2\alpha x_2)$.

$\alpha L(x) = \alpha(x_1 - x_3, 2x_2) = (\alpha x_1 - \alpha x_3, 2\alpha x_2)$.

Thus, $L(\alpha x) = \alpha L(x)$. Therefore, L is a linear transformation.

(2) Let $L: \ P_3 \longrightarrow P_2$, $L(at^3 + bt^2 + ct + d) = (a - b)t^2 + ct$.

(a) Does $2t^3 + 2t^2 + 5 \in \ker \ L$?

Solution: $L(2t^3+2t^2+5) = (2-2)t^2+0\cdot t = 0$ (notice that $a = 2$, $b = 2$, $c = 0$, and $d = 5$, here). Since $L(2t^3 + 2t^2 + 5) = 0$, then $2t^3 + 2t^2 + 5$ is in $\ker \ L$.

(b) Does $2t^2 + 2t + 5 \in \ker \ L$?

Solution: $L(2t^2+2t+5) = (0-2)t^2+2t = -2t^2+2t$ (notice that $a=0$, $b=2$, $c=2$, and $d=5$, here). Since $L(2t^2+2t+5) \neq 0$, then $2t^2+2t+5$ is not in ker L.

(c) Does $5t^2 + 2t \in$ range L?

Solution: Want to find out if the equation $L(at^3+bt^2+ct+d) = 5t^2+2t$ has a solution. I.e. wnat to find out if the equation $(a-b)t^2+ct = 5t^2+2t$ has a solution. But, it's clear from the equation that in order for the equation to have a solution we must have $a-b=5$ and $c=2$. This is a consist system and any polynomial of the form $(b+5)t^3+bt^2+2t+d$ is a preimage for $5t^2+2t$. For example, if you take $b=d=0$, you get $L(5t^3+2t) = 5t^2+2t$.

(d) Does $5t^3 \in$ range L?

Solution: $5t^3 \notin$ range L, because it is not even in the target which is P_2.

(e) Find a basis for ker L.

Solution: ker L consists of all polynomials $p(t) = at^3+bt^2+ct+d$ such that $L(p(t)) = 0$. I.e. all polynomials such that $(a-b)t^2+ct = 0$. This implies $a-b=0$ and $c=0$. Thus, $a=b$, $c=0$, and d is arbitrary. Thus, ker L consists of all polynomials of the form at^3+at^2+d. I.e. all polynomials of the form $a(t^3+t^2)+d(1)$. Thus, $\{t^3+t^2, 1\}$ is a basis for ker L.

(f) Find a basis for range L.

Solution: It's clear from the given formula that the required basis is $\{t^2, t\}$.

(g) Find the matrix of L.

Solution: $L(t^3) = t^2$, $L(t^2) = -t^2$, $L(t) = t$, $L(1) = 0$. Thus, the matrix is

$$\begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(3) $L: \mathbb{R}^3 \longrightarrow \mathbb{R}^2$, $L(x_1, x_2, x_3) = (x_1 - x_3, 2x_2)$

(a) Find a basis for ker L.

Solution: ker L consists of all vectors $x = (x_1, x_2, x_3)$ such that $L(x) = 0$. I.e. all vectors such that $(x_1 - x_3, 2x_2) = 0$. This implies $x_1 - x_3 = 0$ and $2x_2 = 0$. Thus, $x_1 = x_3$ and $x_2 = 0$. Thus, ker L consists of all vectors of the form $(x_1, 0, x_1) = x_1(1, 0, 1)$. Thus, $\{(1, 0, 1)\}$ is a basis for ker L.

(b) Find a basis for range L.

Solution: range L consists of all vectors of the form

$$\begin{bmatrix} x_1 - x_3 \\ 2x_2 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 2 \end{bmatrix}.$$

Since the first two vectors above are linearly dependent, throw away one of them (e.g. the second). Now the remaining two vectors are linearly independent. Thus, the following is a basis:

$$\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \end{bmatrix} \right\}.$$

(c) Find the matrix of L.

Solution: $L(1, 0, 0) = (1, 0)$, $L(0, 1, 0) = (0, 2)$, $L(0, 0, 1) = (-1, 0)$. Thus, the matrix is

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \end{bmatrix}.$$

(4) Find the coordinates of $(3, 4)$ with respect to the basis $[v_1, v_2]$, where $v_1 = (5, 4)$ and $v_2 = (4, 3)$. Also, find the transition matrix from $[e_1, e_2]$ to $[v_1, v_2]$ and the transition matrix from $[v_1, v_2]$ to $[e_1, e_2]$.

Solution: The matrix from $[v_1, v_2]$ to $[e_1, e_2]$ is

$$A = \begin{bmatrix} 5 & 4 \\ 4 & 3 \end{bmatrix}.$$

The matrix from $[e_1, e_2]$ to $[v_1, v_2]$ is

$$A^{-1} = \begin{bmatrix} -3 & 4 \\ 4 & -5 \end{bmatrix}.$$

The required coordinates are:

$$\begin{bmatrix} -3 & 4 \\ 4 & -5 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 7 \\ -8 \end{bmatrix}.$$

(5) Let $v_1 = (5, 4)$, $v_2 = (4, 3)$, $w_1 = (3, 4)$, $w_2 = (5, 7)$. Find the transition matrix corresponding to the change of basis from $[v_1, v_2]$ to $[w_1, w_2]$ and the transition matrix corresponding to the change of basis from $[w_1, w_2]$ to $[v_1, v_2]$.

Solution: Let B be the matrix from $[v_1, v_2]$ to $[e_1, e_2]$ which is $\begin{bmatrix} 5 & 4 \\ 4 & 3 \end{bmatrix}$ and A the matrix from $[e_1, e_2]$ to $[w_1, w_2]$ which is the inverse of the matrix $\begin{bmatrix} 3 & 5 \\ 4 & 7 \end{bmatrix}$; i.e. $A = \begin{bmatrix} 7 & -5 \\ -4 & 3 \end{bmatrix}$. Now the matrix C from $[v_1, v_2]$ to $[w_1, w_2]$ is $C = AB = \begin{bmatrix} 15 & 13 \\ -8 & -7 \end{bmatrix}$.

The matrix from $[w1_1, w_2]$ to $[v_1, v_2]$ is C^{-1}, which is $\begin{bmatrix} 7 & 13 \\ -8 & -15 \end{bmatrix}$.