Assignment 3

Note that one of the questions is not solved yet and I still have to double check those that are solved.

(1) Let A be the matrix given below. Find a basis for the nullspace of A and the nullity of A (i.e. find a basis for the solution space of $Ax = 0$ and also find its dimension). Also, find two bases for the row space of A (one of them should be formed from rows from A and the other is not) and two bases for the column space of A (one of them should be formed from columns from A and the other is not). Find also the row rank of A, the column rank of A, and rank A.

$$A = \begin{bmatrix} 1 & 2 & 2 & -1 & 1 \\ 0 & 2 & 2 & -2 & -1 \\ 2 & 6 & 2 & -4 & 1 \\ 1 & 4 & 0 & -3 & 0 \end{bmatrix}.$$

Solution: The part related to the solution space $Ax = 0$ (nullspace of A), a basis for that, and the dimension of that (the nullity of A) was done as an example in class.

Now find the reduced row echelon form of A, which is

$$B = \begin{bmatrix} 1 & 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & -1 & -1/2 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The nonzero rows of B are basis for the row space of A. I.e.

$$\{[1\ 0\ 0\ 1\ 2],\ [0\ 1\ 0\ -1\ -1/2],\ [0\ 0\ 1\ 0\ 0]\}$$

is basis for the row space of A that does not contain rows from A. Thus, the row rank of A is 3, and hence, rank $A = 3$.

Take the columns (make them as column vectors) of A corresponding to the columns of B with the leading 1's (these are columns 1, 2, and 3), which are

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 6 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 2 \\ 0 \end{bmatrix} \right\}.$$

These vectors form a basis for the columnspace of A that is formed from columns from A. Thus, the column rank of A is 3.

Now find the reduced row echelon form of A^T, which is

$$C = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The nonzero rows of C (made as columns) are basis for the column space of A. I.e.

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}$$

are basis for the column space of A which does not contain columns from A.

Take the columns (make them as row vectors) ofA^T corresponding to the columns of C with the leading 1's (these are columns 1, 2, and 3), which are

$$\{[1\ 2\ 2\ -1\ 1],\ [0\ 2\ 2\ -2\ -1],\ [2\ 6\ 2\ -4\ 1]\}.$$

These vectors form a basis for the rowspace of A that is formed from rows of A.

(2) Let $v_1 = (2, -1, 1)$ and $v_2 = (-3, 1, 7)$.

(a) Find a nonzero vector in $\mathbb{R}^3$ that is orthogonal to v_1 and v_2. Call this vector v_3.

Solution: Want $v_3 = (a, b, c)$ to satisfy $v_3 \cdot v_1 = 0$ and $v_3 \cdot v_2 = 0$. Thus, we get a system of two equations: $2a - b + c = 0$ and $-3a + b + 7c = 0$. Now solve the system to get: $(8s, 17s, s)$ is a solution, where s is any real number. We don't want v_3 to be zero. Thus, take s to be any nonzero real number; e.g. take $s = 1$, to get $v_3 = (8, 17, 1)$.

(b) Now let $S = \{v_1, v_2, v_3\}$. Then make S orthonormal and then write the vector $v = (-6, 7, 8)$ as a linear combination of the *new* vectors in S.

Solution: To make them orthonormal, divide each vector by its norm. Note that $\|v_1\| = \sqrt{6}$, $\|v_2\| = \sqrt{59}$, and $\|v_3\| = \sqrt{354}$. Call the new vectors w_1, w_2 and w_3 respectively.

Now we need to find c_1, c_2 and c_3 such that $v = c_1w_1 + c_2w_2 + c_3w_3$. Do the procedure we did in class to get $c_i = v \cdot w_i$, $i = 1, 2, 3$. Thus, $c_1 = -11/\sqrt{6}$, $c_2 = 81/\sqrt{59}$, and $c_3 = 79/\sqrt{354}$.

(c) Is it possible to have an orthonormal set of 7 vectors in $\mathbb{R}^6$? Explain.

Solution: No. because every orthonormal set is linearly independent. So, if we have an orthonormal set of 7 vectors in $\mathbb{R}^6$, then we'll have a linearly independent set of 7 vectors in $\mathbb{R}^6$. But, that can't be because dim $\mathbb{R}^6 = 6$, which means any linearly independent set in $\mathbb{R}^6$ can have at most 6 linearly independent vectors.

(3) Question 2 Section 6.8 (of the text).

(4) Tranform the following basis S for $\mathbb{R}^3$ to an orthonormal basis using Gram-Schmidt process

$$S = \left\{ u_1 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix},\ u_2 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix},\ u_3 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \right\}.$$

Solution: Let $v_1 = u_1$, $v_2 = u_2 - \frac{u_2 \cdot v_1}{v_1 \cdot v_1} v_1 = \begin{bmatrix} 4/3 \\ -1/3 \\ 5/3 \end{bmatrix}$, and $v_3 = u_3 - \frac{u_3 \cdot v_1}{v_1 \cdot v_1} v_1 - \frac{u_3 \cdot v_2}{v_2 \cdot v_2} v_2 = \begin{bmatrix} -1/7 \\ -3/14 \\ 1/14 \end{bmatrix}$.

Now let $w_i = \frac{v_i}{\|v_i\|}$, $i = 1,\ 2,\ 3$. Then $\{w_1, w_2, w_3\}$ is an orthonormal basis for $\mathbb{R}^3$.