Answers for Assignment 2

(1) Determine if the each of the following sets of vectors form a basis for $\mathbb{R}^3$. Give full explanation. If the given set is a basis for $\mathbb{R}^3$, write the vector $(2,3,4)$ as a linear combination of the vectors in the basis:

(a) $\left\{ \begin{bmatrix} 3 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} \right\}$.

Solution: Let's call the three vectors (in order) v_1, v_2, and v_3. We have 3 vectors which is the same as $\dim \mathbb{R}^3$. So, it suffices to check if they are linearly independent (if yes, then they form a basis for $\mathbb{R}^3$; if not, then they don't). In order for the given vectors to be linearly independent, the system $c_1v_1 + c_2v_2 + c_3v_3 = 0$ must have only the trivial solution. Solve the system

$$\begin{bmatrix} 3 & -1 & 3 \\ 2 & 2 & 0 \\ 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

by either Gaussian elimination or Gauss-Jordan elimination or the inverse method to find out that the system has only the trivial solution which implies the given vectors are linearly independent, and hence, they form a basis for $\mathbb{R}^3$. Equivalently to check for linear independence, you can use the determinant test (you can do that because the matrix of coefficients here is a square matrix). It's easy to see that the determinant of the matrix of coefficients is 2. Since it's nonzero, then the given vectors are linearly independent.

Now to write $(2,3,4)$ as a linear combination of the given vectors, solve the system

$$\begin{bmatrix} 3 & -1 & 3 \\ 2 & 2 & 0 \\ 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$$

by either Gaussian elimination or Gauss-Jordan elimination or the inverse method to find out that $c_1 = -4$, $c_2 = 11/2$, and $c_3 = 13/2$.

(b) $\left\{ \begin{bmatrix} 3 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right\}$.

Solution: Here the number of given vectors is greater than dim $\mathbb{R}^3$ which is 3, and hence the given vectors are linearly dependent, and therefore not a basis for $\mathbb{R}^3$.

(c) $\left\{ \begin{bmatrix} 3 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 5 \end{bmatrix} \right\}$.

Solution: Here the number of given vectors is less than dim $\mathbb{R}^3$ which is 3, and hence the given vectors do not span $\mathbb{R}^3$, and therefore not a basis for $\mathbb{R}^3$.

(d) $\left\{ \begin{bmatrix} 3 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \\ 6 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$.

Solution: The given vectors contain the zero vector. Thus, they are linearly dependent, and therefore not a basis for $\mathbb{R}^3$.

(2) Determine if the following set of polynomials form a basis for P_2. Give full explanation. If the given set is a basis for P_2, write the polynomial $2+3t+4t^2$ as a linear combination of the polynomials in the basis:

$$\left\{3 + 2t + 2t^2,\ -1 + 2t + t^2,\ 3 + t^2\right\}$$

Solution: Since the number of the given polynomials is equal to the dimension of P_2, it suffices to check if they are linearly independent (if yes, then they form a basis for P_2; if not, then not a basis). In order for the given vectors (call them v_1, v_2, and v_3, respectively) to be linearly independent, the system $c_1v_1 + c_2v_2 + c_3v_3 = 0$ must have only the trivial solution. Now since the system you get here is the same as that in (1)(a), you can use your work there to get that the given vectors are linearly independent, and hence, they form a basis for P_2, and $2 + 3t + 4t^2 = -4v_1 + (11/2)v_2 + (13/2)v_3$.

(3) Decide if each of the following sets of vectors spans $\mathbb{R}^3$. Give full explanation. If the given set does not span $\mathbb{R}^3$, give an example of a vector in $\mathbb{R}^3$ that does not belong to the span of the given set:

(a) $\left\{ \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 5 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 5 \\ 1 \end{bmatrix} \right\}.$

Solution: In order for the given vectors (call them v_1, v_2, v_3, and v_4, respectively) to span $\mathbb{R}^3$, the system $c_1v_1 + c_2v_2 + c_3v_3 + c_4v_4 = v$ must have a solution for every vector $v = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ in $\mathbb{R}^3$. The system can be written as

$$\begin{bmatrix} -1 & 2 & 0 & 0 \\ 2 & 1 & 5 & 5 \\ 1 & -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}.$$

Now perform Gaussian elimination or Gauss-Jordan elimination on the system. If you do Gaussian elimination, you'll get the following row echelon form (the operations I did are - in order - as follows: $r_1 \longleftrightarrow r_3$, $r_2 - 2r_1 \longrightarrow r_2$, $r_3 + r_1 \longrightarrow r_3$, $r_2 \longleftrightarrow r_3$, $r_3 - 3r_2 \longrightarrow r_3$, $r_3/10 \longrightarrow r_3$).

$$\begin{bmatrix} 1 & -1 & -1 & 1 & c \\ 0 & 1 & -1 & 1 & a+c \\ 0 & 0 & 1 & 0 & (b-3a-5c)/10 \end{bmatrix}$$

. It's clear the system is consistent, and hence it has a solution. Therefore, the given vectors span $\mathbb{R}^3$.

(b) $\left\{ \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 5 \\ 1 \end{bmatrix}. \right\}$

Solution: In order for the given vectors (call them v_1, v_2, and v_3, respectively) to span $\mathbb{R}^3$, the system $c_1v_1 + c_2v_2 + c_3v_3 = v$ must have a solution for every vector $v = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ in $\mathbb{R}^3$. The system can be written as

$$\begin{bmatrix} -1 & 2 & 0 \\ 2 & 1 & 5 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}.$$

Now perform Gaussian elimination or Gauss-Jordan elimination on the system. If you do Gaussian elimination, you'll get the following row echelon form (the operations I did are - in order - as follows: $r_1 \longleftrightarrow r_3$, $r_2 - 2r_1 \longrightarrow r_2$, $r_3 + r_1 \longrightarrow r_3$, $r_2 \longleftrightarrow r_3$, $r_3 - 3r_2 \longrightarrow r_3$, which are the same as those in the previous part except that the last operation in the previous part was not needed here.)

$$\begin{bmatrix} 1 & -1 & 1 & c \\ 0 & 1 & 1 & a + c \\ 0 & 0 & 0 & (b - 3a - 5c) \end{bmatrix}.$$

It's clear the system is inconsistent when $b - 32 - 5c \neq 0$. Therefore, the given vectors do not span $\mathbb{R}^3$. For example, if you take $b = 1$, $a = c = 0$, then the vector $(0, 1, 0)$ is not in the span of v_1, v_2, and v_3.

(4) Deteremine if the given set of polynomials is linearly independent in P_2. Give full explanation. If the set is linearly dependent, write one of the polynomials in the set as a linear combination of the other polynomials in the set.

$$\left\{3 + 2t + 2t^2,\ -1 + 2t + t^2,\ 11 + 2t + 4t^2\right\}$$

Solution: Call the given vectors v_1, v_2, and v_3. If the system $c_1v_1 + c_2v_2 + c_3v_3 = 0$ has only the trivial solution, then the vectors are linearly independent. If the system has a nontrivial solution, then the vectors are linearly dependent. Solve the

$$\begin{bmatrix} 3 & -1 & 11 \\ 2 & 2 & 2 \\ 2 & 1 & 4 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

by either Gaussian elimination or Gauss-Jordan elimination or the inverse method to find out that the system has a non-trivial solution, and hence, the given vectors are linearly dependent. If you perform the elementary row operations $r_1 \longleftrightarrow r_2$, $r_1/2 \longrightarrow r_1$, $r_2 - 3r_1 \longrightarrow r_2$, $r_3 - 2r_1 \longrightarrow r_3$, $r_2/-4 \longrightarrow r_2$, $r_2 + r_3 \longrightarrow r_3$, you'll get the following row echelon form:

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Solve, to get c_3 is arbitrary, $c_2 = 2c_3$, and $c_1 = -3c_3$. Thus, the system has nontrivial solutions. In particular, if you take $c_3 = 1$, then $c_2 = 2$, and $c_1 = -3$. Now since $c_1v_1 + c_2v_2 + c_3v_3 = 0$, then $-3v_1 + 2v_2 + v_3 = 0$. Hence, $v_3 = 3v_1 - 2v_2$. Thus, we wrote v_3 as a linear combination of v_1 and v_2.

(5) Decide if the following set S is a subspace of the given vector space V:

(a) $V = \mathbb{R}^4$. S is the set of all vectors in V of the form (a, b, c, d), where $c = 3$ and $d = a - 3b$.

Solution: It's clear the given set is not closed under vector addition and under scalar multiplication (e.g. $(0, 0, 3, 0) \in S$, but $(0, 0, 3, 0) + (0, 0, 3, 0) = (0, 0, 6, 0) \notin S$) and also not closed under scalar multiplication (e.g. $(0, 0, 3, 0) \in S$, but $2(0, 0, 3, 0) = (0, 0, 6, 0) \notin S$). Hence, S is not a subspace of V. Note that you can just say the zero vector is not in S and hence it cannot be a subspace (note that if the zero vector is in S, that does not necessarily mean S is a subspace).

(b) $V = P_2$. S is the set of all polynomials in V of the form $a + bt + ct^2$, where $a = b + c$.

Solution: S consists of all polynomials of the form $b + c + bt + ct^2$; i.e. all polynomials in P_2 such that the constant term is equal to the sum of the coefficient of t and the coefficient of t^2. It is easy to see that S is a subspace of P_2. To do that, you need to prove S is closed under vector addition and under scalar multiplication. To show S is closed under vector addition, let p_1 and p_2 be in S. Then we can write $p_1 = b_1 + c_1 + b_1t + c_1t^2$ and $p_2 = b_2 + c_2 + b_2t + c_2t^2$. Now $p_1 + p_2 = (b_1+b_2+c_1+c_2)+(b_1+b_2)t+(c_1+c_2)t^2$, which is in S. To show S is closed under scalar multiplication, let α be a scalar, and $p_1 \in S$. Then we can write $p_1 = b_1 + c_1 + b_1 t + c_1 t^2$. Thus, $\alpha p_1 = \alpha(b_1 + c_1) + \alpha b_1 t + \alpha c_1 t^2$, which is in S.

(6) Find a basis for $\mathbb{R}^3$ that includes the vectors $\left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$.

Solution: Perform the elementary row operations $r_3 - 2r_1 \longrightarrow r_3$, $r_1 - r_2 \longrightarrow r_1$, $r_3 + r_2 \longrightarrow r_3$, $r_3/-2 \longrightarrow r_3$, $r_1 - r_3 \longrightarrow r_1$, on the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 2 & 1 & 0 & 0 & 1 \end{bmatrix}$$

(note that A is formed from the given vectors and the standard basis for $\mathbb{R}^3$; i.e. it's formed from the given vectors and the 3×3 identity matrix) to get the matrix

$$B = \begin{bmatrix} 1 & 0 & 0 & -1/2 & 1/2 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1/2 & -1/2 \end{bmatrix}.$$

Now take the columns of A corresponding to the columns of B with the leading 1's, to get:

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$$

is a basis for $\mathbb{R}^3$ that contains the given vectors.

(7) Let $S = \{-1 + 2t + t^2, 2 + t - t^2, 5t - t^2, 5t + t^2\}$. Find a basis from S for span S.

Solution: Call the given polynomials (in order) v_1, v_2, v_3, and v_4. Perform the elementary row operations $r_1 \longleftrightarrow r_3$, $r_2 - 2r_1 \longrightarrow r_2$, $r_3 + r_1 \longrightarrow r_3$, $r_2 \longleftrightarrow r_3$, $r_1 + r_2 \longrightarrow r_1$, $r_3 - 3r_2 \longrightarrow r_3$, $r_3/10 \longrightarrow r_3$, $r_1 + 2r_3 \longrightarrow r_1$, $r_2 + r_3 \longrightarrow r_2$, on the matrix

$$A = \begin{bmatrix} -1 & 2 & 0 & 0 \\ 2 & 1 & 5 & 5 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

(note that A is formed from the coefficients of the given polynomials) to get the matrix

$$B = \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

Now take the columns of A corresponding to the columns of B with the leading 1's, to get:

$$\{v_1, v_2, v_3\}$$

is a basis for span S from S.

(8) Find a basis in M_{33} for all 3×3 skew-symmetric matrices.

Solution: A 3×3 skew-symmetric matrix has the form

$$A = \begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix}.$$

Now seperate the variables in the previous matrix (the variables are a, b, and c) to get

$$A = \begin{bmatrix} 0 & a & 0 \\ -a & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & b \\ 0 & 0 & 0 \\ -b & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & c \\ 0 & -c & 0 \end{bmatrix}.$$

Now factor the variables out to get

$$A = a \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}.$$

Thus, the above three matrices span the subspace consisting of the 3×3 skew-symmetric matrices. Now it's easy to verify that the above three matrices are linearly independent. Thus, the required basis is

$$\left\{ \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \right\}.$$

(9) Find a basis for the subspace of $\mathbb{R}^3$ consisting of all vectors of the form $v = \begin{bmatrix} 2b-a \\ 2a+b+5c \\ a-b+c \end{bmatrix}$.

Solution: As in the previous example, separate the variables to get

$$v = \begin{bmatrix} -a \\ 2a \\ a \end{bmatrix} + \begin{bmatrix} 2b \\ b \\ -b \end{bmatrix} + \begin{bmatrix} 0 \\ 5c \\ c \end{bmatrix}.$$

Now factor the variables out to get:

$$v = a\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} + b\begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} + c\begin{bmatrix} 0 \\ 5 \\ 1 \end{bmatrix}.$$

Thus, the set

$$S = \left\{ v_1 = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}, v_3 = \begin{bmatrix} 0 \\ 5 \\ 1 \end{bmatrix} \right\}$$

spans the given subspace. Now you need to check whether the set is linearly independent or not (you know how to do that, I'll leave it to you to do that). It's clear that it's linearly dependent. Now find a basis from S for span S using the method of Question 7 (or you can say $v_3 = 2v_1 + v_2$, so remove v_3; then say v_1 and v_2 are linearly independent, because neither one of them is a multiple of the other). When you do that, you'll find out that $S = \{v_1, v_2\}$ is a basis for the given subspace.