Complex Numbers

The standard form of a complex number is $a + bi$, where a and b are real numbers, and $i = \sqrt{-1}$. "a" is called the real part of the complex number and "bi" is called the imaginary part. *Remarks:*

(1) Since $i = \sqrt{-1}$, it follows that $i^2 = -1$. Similarly, $i^3 = i.i^2 = i.(-1) = -i$, $i^4 = i^2.i^2 = (-1).(-1) = 1$, $i^5 = i.i^4 = i.1 = i$, $i^6 = i.i^5 = i.i = i^2 = -1$, etc.

(2) If n is a positive real number, then $\sqrt{-n} = \sqrt{n}.\sqrt{-1} = \sqrt{n}.i$.

For example, $\sqrt{-4} = 2i$, $\sqrt{-5} = \sqrt{5}i$, $\sqrt{\frac{-1}{4}} = \frac{1}{2}i$.

Now let $a + bi$ and $c + di$ be complex numbers, then

- $a + bi = c + di$ if and only if $a = c$ and $b = d$.
- $(a + bi) + (c + di) = (a + c) + (b + d)i$.
- $(a + bi) - (c + di) = (a - c) + (b - d)i$.
- $(a + bi)(c + di) = (ac - bd) + (ad + bc)i$.

- $\frac{a+bi}{c+di} = \frac{a+bi}{c+di}.\frac{c-di}{c-di} = \frac{(a+bi)(c-di)}{(c+di)(c-di)} = \frac{(ac+bd)+(bc-ad)i}{c^2+d^2} = \frac{ac+bd}{c^2+d^2} + \frac{bc-ad}{c^2+d^2}i$

Remark: The *conjugate* of the complex number $a + bi$, is written as $\overline{a + bi}$, and is defined to be $\overline{a + bi} = a - bi$. Thus, to simplify any complex fraction, we need first to multiply both the numerator and the senominator by the conjugate of the denominator and then we need to simplify.

EXAMPLES:

(1) $i^{49} = i.i^{48} = i.(i^4)^{12} = i.1^{12} = i.1 = i$.

(2) $(2 + 3i) - (4 - 7i) = (2 - 4) + (3 - (-7))i = -2 + 10i$.

(3) $(2 + 3i) + (4 - 7i) = (2 + 4) + (3 - 7)i = 6 - 4i$.

(4) $(2 + 3i)(4 - 7i) = (2)(4) + (2)(-7i) + (3i)(4) + (3i)(-7i) = 8 - 14i + 12i - 21i^2 = 8 - 2i - (21)(-1) = (8 + 21) - 2i = 29 - 2i$.

(5) $\frac{2+3i}{4-7i} = \frac{2+3i}{4-7i}.\frac{4+7i}{4+7i} = \frac{(2+3i)(4+7i)}{(4-7i)(4+7i)} = \frac{(2)(4)+(2)(7i)+(3i)(4)+(3i)(7i)}{4^2+7^2} = \frac{8+14i+12i+21i^2}{15+49} = \frac{8+26i+(21)(-1)}{64} = \frac{8-21+26i}{64} = \frac{-13+26i}{64} = \frac{-13}{64} + \frac{26}{64}i = \frac{-13}{64} + \frac{13}{32}i$.

(6) $2i(3 - 2i) = (2i)(3) - (2i)(2i) = 6i - 4i^2 = 6i - 4(-1) = 6i + 4 = 4 + 6i$.

(7) $\frac{2-3i}{-3i} = \frac{2-3i}{-3i}.\frac{3i}{3i} = \frac{(2-3i)(3i)}{(-3i)(3i)} = \frac{2(3i)-(3i)(3i)}{-9i^2} = \frac{6i-9i^2}{(-9)(-1)} = \frac{6i-9(-1)}{9} = \frac{9+6i}{9} = \frac{9}{9} + \frac{6}{9}i = 1 + \frac{2}{3}i$.

Remark: $(a+bi)(a-bi) = a^2+b^2$. For example, $(-2+3i)(-2-3i) = (-2)^2+3^2 = 4+9 = 13$.

More Examples:

(1) Simplify $(2-i)^2$ and express the answer in the standard form (i.e., as $a+bi$).
Solution: $(2-i)^2 = (2-i)(2-i) = (2)(2)+(2)(-i)-(i)(2)-(i)(-i) = 4-2i-2i+i^2 = 4-4i+(-1) = 3-4i$.

(2) Simplify $i^{27}-3i^7+4i^8-5i^{10}+7-i$ and express the answer in the standard form.
Solution: Notice first that $i^{27} = i^{24}.i^3 = (i^4)^6.i^2.i = 1^6.(-1).i = 1.(-1).i = -i$.
Similarly, $i^7 = i^4.i^3 = 1.i^3 = i^3 = i^2.i = (-1).i = -i$.
$i^8 = (i^4)^2 = 1^2 = 1$.
$i^{10} = i^8.i^2 = 1.i^2 = i^2 = -1$.
So, the above expression is equal to:
$-i-3.(-i)+4.(1)-5.(-1)+7-i = -i+3i+4+5+7-i = 2i+9+7-i = 16+i$.

(3) Solve the equation: $4x^2+9=0$. Express the answers in the standard form.
Solution: $4x^2=-9 \Longrightarrow x^2 = \frac{-9}{4} \Longrightarrow x = \pm\sqrt{\frac{-9}{4}} = \pm\sqrt{\frac{9}{4}}.i = \pm\frac{3}{2}i$.

(4) Solve the equation: $-4x^2+2x-1=0$. Express the answers in the standard form.
Solution: Use the quadratic formula. Here, $a=-4$, $b=2$, $c=-1$. So, the solutions, are:
$x = \frac{-(2)\pm\sqrt{2^2-(4)(-4)(-1)}}{(2)(-4)} = \frac{-2\pm\sqrt{-12}}{-8} = \frac{-2\pm\sqrt{12}i}{-8} = \frac{-2}{-8} \pm \frac{\sqrt{12}}{-8}i = \frac{1}{4} \pm \frac{\sqrt{12}}{-8}i$.
You can stop here, but the answer can be further simplified, as follows:
$\frac{1}{4} \pm \frac{\sqrt{3}}{4}i$.

Remarks: Let $z_1 = a_1+b_1i$ and $z_2 = a_2+b_2i$ be complex numbers. Then

(1) The magnitude/length of z_1, denoted $|z_1$, is $|z_1| = \sqrt{z\overline{z}} = \sqrt{a_1^2+b_1^2}$.
(2) The complex number z_1 is sometimes represented as the ordered pair (a_1, b_1).
(3) The complex plane is drawn as the Cartesian plan except that the x-axis is called the real axis (denoted Re), and the y-axis is called the imaginary axis (denoted Im). We discussed how to draw z_1 in the comlpex plane in class.
(4) $z_2 = z_1$ if and only if $a_2 = a_1$ and $b_2 = b_1$.
(5) $z_1 = 0$ if and only if $a_1 = 0$ and $b_1 = 0$.
(6) $|z_1z_2| = |z_1||z_2|$.
(7) If $z_2 \neq 0$, then $|z_1/z_2| = |z_1|/|z_2|$.
(8) $|z_1|^2 = z_1\overline{z_1}$.
(9) In some fields such as physics and engineering, some people use j instead of i to denote $\sqrt{-1}$.

Complex Matrices

Let $C = (c_{ij})$ be an $n \times n$ complex matrix. Then

(1) C can be written as $C = A+iB$, where A and B are real-valued matrices (i.e. the entries of A and B are real).
(2) The *conjugate* of C, denoted $\overline{C}$, is $A-iB$ (i.e. take the conjugate of every element of C).
(3) Thus, $\overline{C} = (\overline{c_{ij}})$.
(4) The Hermitian transpose of C, denoted C^H or C^*, is $C^H = (\overline{C})^T = \overline{(C^T)}$.