Boolean Expressions and NP-complete Problems

Definition 1. A *boolean expression* is an expression built from:

- The binary operators $\vee$, $\wedge$.
- The unary operator $\sim$.
- Parantheses.

Remark 0.1. Notice that the priority is for $\sim$, then $\wedge$, then $\vee$.

Note: We'll use 1 for true and 0 for false.

Notation: We'll use BE for *boolean expression*.

Definition 2. Let E be a BE. A *truth assignment* for E is an assignment that assigns either true or false to each one of the variables of E.

Notation: We'll use TA for *truth assignment*.

Notation: If T is a TA for a boolean expression E, then we'll use $E(T)$ for the value of E corresponding to T.

Definition 3. A truth assignment T *satisfies* a boolean expression E if $E(T) = 1$.

Definition 4. A boolean express expression E is said to be *satisfiable* if there exists at least one TA that satsifies E.

Example 5. Let $E = x_1 \wedge \sim (x_2 \vee x_3)$. Then E is satisfiable. Consider the truth assignment T defined by: $T(x_1) = 1$, $T(x_2) = 0$, $T(x_3) = 0$. Then $E(T) = 1$.

Example 6. $E = x_1 \wedge (\sim x_1 \vee x_2) \wedge (\sim x_2)$ is not satisfiable, because in order for a truth assignment T to satisfy E, we need $T(x_1) = 1$ and $T(x_2) = 0$. But, if so, then $\sim x_1 \vee x_2$ will be false. Hence, $E(T) = 0$.

Definition 7. The *satisfiability problem*, denoted by SAT is the problem:

Given a BE, is it satisfiable?

THEOREM 8. *(Cook's Theorem) SAT is NP-complete.*

DEFINITION 9. A *literal* is a variable or a negated variable (e.g. x_1, $\sim x_2$).

DEFINITION 10. A *clause* is one or more literals combined by $\vee$.

EXAMPLE 11. The following are clauses:

- x_1.
- $x_1 \vee \sim x_2$.
- $x_1 \vee x_2 \vee x_3$.

DEFINITION 12. A BE is said to be in *conjunctive normal form (CNF)* if it consists of clauses combined by $\wedge$.

EXAMPLE 13. $x_1 \wedge (x_1 \vee x_2) \wedge (\sim x_1 \vee x_2 \vee x_3)$ is a BE in CNF.

DEFINITION 14. A BE is said to be in *k-conjunctive normal form (k-CNF)* if it is in CNF and if each clause has exactly k distinct literals.

EXAMPLE 15. $(x_1 \vee x_2 \vee \sim x_3) \wedge (x_1 \vee \sim x_2 \vee x_3)$ is in $3 - CNF$.

DEFINITION 16. The $CSAT$ problem is the problem:

Given a BE in CNF, is it satisfiable?

DEFINITION 17. The $k - SAT$ problem is the problem:

Given a BE in $k - CNF$, is it satisfiable?

Note: Sometimes we'll write $kSAT$ instead of $k - SAT$.

THEOREM 18. *$3SAT$ is NP-complete.*

DEFINITION 19. The *$CLIQUE$ problem* is the problem:

Given an undirected graph G and a positive integer k, does G have a clique of size k.

Note: Sometimes the CLIQUE problem is stated as follows:

CLIQUE=$\{\langle G, k\rangle$: G has a clique of size $k\}$.

THEOREM 20. *CLIQUE is NP-complete.*

PROOF. The proof was give in class. It was done in two steps.

- We proved that CLIQUE is NP.
- We reduced in polynomial time $3SAT$ to CLIQUE.

□

DEFINITION 21. The *VERTEX-COVER problem* is the problem:

Given an undirected graph G and a positive integer k, does G have a vertex cover of size k.

Note: Sometimes the VERTEX-COVER problem is stated as follows:

VERTEX-COVER=$\{\langle G, k\rangle$: G has a vertex cover of size $k\}$.

THEOREM 22. *VERTEX-COVER is NP-complete.*

PROOF. The proof was give in class. It was done in two steps.

- We proved that VERTEX-COVER is NP.
- We reduced in polynomial time CLIQUE to VERTEX-COVER.

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