

Questions about the Pigeonhole Principle and Derangements

1. Prove that D_n is even only if n is odd.

Solution: First write D_n in the recursive form $D_n = nD_{n-1} + (-1)^n$. The proof is by cases:

Case 1: n is even. In this case, $(-1)^n = 1$ and nD_{n-1} is even. So, D_n being a sum of an even number (nD_{n-1}) and an odd number (1), is odd.

Case 2: n is odd. In this case, $(-1)^n = -1$ and nD_{n-1} is odd. Notice here that $n-1$ is even and so by case 1, D_{n-1} is odd. Thus, nD_{n-1} being a product of two odd numbers is odd. So, D_n being a sum of an odd number (nD_{n-1}) and another odd number (-1), is even.

2. Find a single value for the expression $\sum_{k=0}^n C(n, k)D_{n-k}$. Notice that $D_0 = 1$.

Solution: $n!$.

3. 300 people are to sit on 300 seats numbered from 1 to 300.

- In how many ways can those people sit.

Solution: $300!$.

- Suppose now that all of those people decided to change their seats. In how many ways can they sit now?

Solution: D_{300} .

4. Prove that if $n+1$ (*distinct*) integers are chosen from the set $\{1, 2, \dots, 2n\}$, then two of the chosen numbers differ by 1.

Solution: Let the $n+1$ chosen numbers be $c_1, c_2, \dots, c_{n+1}$. Now consider $c_1 + 1, c_2 + 1, \dots, c_{n+1} + 1$. We have a total of $2n+2$ numbers whose values are in the set $\{1, 2, \dots, 2n+1\}$. Thus, we have to put $2n+2$ objects in $2n+1$ boxes. Therefore, at least two of the objects must be in the same box. This means that two of the above numbers must be equal. But, all numbers in the set $\{c_1, c_2, \dots, c_{n+1}\}$ are distinct and so are the numbers in the set $\{c_1 + 1, c_2 + 1, \dots, c_{n+1} + 1\}$. Thus, it must be that $c_i = c_j + 1$, for some $1 \leq i \leq n+1$, $1 \leq j \leq n+1$. Thus, $c_i - c_j = 1$.

5. What is the minimum number of people in a city of 2 millions who share the same birth month and the same first and second initials?

Solution: The number of possible combinations of birth month and first and second initials is $(12)(26)(26)$. Thus, we want to put 2,000,000 objects into $(12)(26)(26)$. Thus, at least $\lceil \frac{2000000}{(12)(26)(26)} \rceil$ have to be in one box. So, the answer is 247.

6. A college offers 521 classes. Classes can be held at 20 different times. What is the minimum number of classrooms needed?

Solution: 27.

7. A person has to choose 51 integers. The minimum has to be 1000 and the maximum 1099. Is it possible for that person to choose those 51 numbers so that no two of them are consecutive?

Solution: Similar to 4.

8. Show that if 5 integers are chosen from the set $\{1, 2, \dots, 8\}$, then two of those 5 integers must add up to 9.
9. Suppose that $k_1 + k_2 + \dots + k_n - n + 1$ objects are put into n boxes, where $k_i, i = 1, \dots, n$ are positive integers and so is n . Prove that either the first box contains at least k_1 objects, or the second contains at least k_2 objects, ..., or the n th box contains at least k_n objects.

Solution: Assume the first box contains less than k_1 , the second less than k_2 , ..., the n th less than k_n . Then the total number of objects in the n boxes is less than or equal to $(k_1 - 1) + (k_2 - 1) + \dots + (k_n - 1) = k_1 + k_2 + \dots + k_n - n$ which is less than the number of objects we have. This is a contradiction.

10. 17 persons have first names Chris, Eric, and Mike and last names Kincaid and Smith. Show that at least 3 persons among those have the same first and last names and at least 6 of them have the same first name.
11. An inventory consists of a list of 115 items, each marked "available" or "unavailable". There are 60 available items. Show that there are at least 2 available items in the list exactly 4 items apart.
12. Don't forget the examples I gave in class today and the next group of questions. More questions will be added regularly.