CSCE 235, Spring 2001

Homework 4 Solutions

Question 1: Solve the following recurrence relation:

$$\frac{\sqrt{a_n}}{a_{n-1}} = a_{n-2}^4,\ n \geq 2,\ a_0 = 1,\ a_1 = 4.$$

SOLUTION:

Take $\log_2$ of both sides, to get

$$\frac{1}{2}\mathbf{log}_2 a_n - \mathbf{log}_2 a_{n-1} - 4\mathbf{log}_2 a_{n-2} = 0.$$

Now let $b_n = \mathbf{log}_2 a_n$, to get

$$\frac{1}{2}b_n - b_{n-1} - 4b_{n-2} = 0,\ \ b_0 = 0,\ \ b_1 = 2.$$

Thus,

$$b_n - 2b_{n-1} - 8b_{n-2} = 0,\ \ b_0 = 0,\ \ b_1 = 2.$$

The characterestic equation is: $x^2 - 2x - 8 = 0$, which has solutions 4 and -2. Thus,

$$b_n = \beta_1 \cdot 4^n + \beta_2(-2)^n.$$

Now substite the initial conditions to get $\beta_1 = \frac{1}{3}$ and $\beta_2 = -\frac{1}{3}$. Thus,

$$b_n = \frac{1}{3} \cdot \left[4^n - (-2)^n\right],\ \ n \geq 0.$$

Therefore,

$$a_n = 2^{\frac{1}{3}\cdot\left[4^n - (-2)^n\right]},\ \ n \geq 0.$$

Question 2:

Definition: Let $G = (V, E)$ be a simple undircted graph. Then, the complement of G, denoted $\overline{G}$, is the simple undirected graph whose vertex set is V and whose edge set is

$$\overline{E} = \{(u, v) \mid u \in V,\ v \in V,\ (u, v) \notin E\}.$$

In other words, $\overline{G}$ is a graph with the same vertex set as G and with an edge set of the missing edges (*only*) of G.

The following parts are unrelated. n and m are natural numbers.

1. What is the degree sequence of $K_{m,n}$?

   SOLUTION:

$$\begin{cases} \underbrace{m, m, ..., m}_{n \textbf{ times}}, \underbrace{n, n, ..., n}_{m \textbf{ times}} & \textbf{if } m \geq n \\ \underbrace{n, n, ..., n}_{m \textbf{ times}}, \underbrace{m, m, ..., m}_{n \textbf{ times}} & \textbf{if } m < n \end{cases}$$

2. How many edges does $\overline{K_{m,n}}$ have?

   SOLUTION:

$$\begin{cases} \frac{(m+n)(m+n-1)}{2} - m \cdot n & \textbf{if } m + n \geq 3 \\ 0 & \textbf{if } m = n = 1 \end{cases}$$

3. What is the degree sequence of $\overline{K_{m,n}}$?

   SOLUTION: **If** $m = n = 1$**, then the degree sequence is 0,0. If** $m + n \geq 3$**, then it is**

$$\begin{cases} \underbrace{m-1, m-1, ..., m-1}_{m \textbf{ times}}, \underbrace{n-1, n-1, ..., n-1}_{n \textbf{ times}} & \textbf{if } m \geq n \\ \underbrace{n-1, n-1, ..., n-1}_{n \textbf{ times}}, \underbrace{m-1, m-1, ..., m-1}_{m \textbf{ times}} & \textbf{if } m < n \end{cases}$$

4. How many edges does $\overline{C_n}$ have?

   SOLUTION:

$$\begin{cases} \frac{n\cdot(n-1)}{2} - n & \textbf{if } n \geq 3 \\ 0 & \textbf{if } n = 1, 2 \end{cases}$$

5. What is the degree sequence of $\overline{C_n}$?

   SOLUTION:

$$\begin{cases} 0 & \textbf{if } n = 1 \\ 0, 0 & \textbf{if } n = 2 \\ \underbrace{n-3, n-3, ..., n-3}_{n \textbf{ times}} & \textbf{if } n \geq 3 \end{cases}$$

6. Is there a simple undirected graph with a degree sequence 5, 2, 1, 1, 1?

   SOLUTION: **No, because the maximum vertex-degree in a simple undirected graph of 5 vertices is 4, and so we can't have 5 in the degree sequence.**

7. Let G be a simple undirected graph. Is it possible for both G and $\overline{G}$ to have Euler cycles? If yes, write down an example.

   SOLUTION: **Yes, consider C_5 and $\overline{C_5}$. Both of them are connected and all vertices in each one of them have even degrees (actually these two graphs are isomorphic and so the result follows directly from this fact).**

8. Prove or disprove the following:

   If i and j are natural numbers, with $i < j$ and $i \geq 300$, then C_i is a subgraph of C_j.

Warning: *Do not consider particular values of n.*

Remark: *Denote the vertex set of C_i by $V = \{1, 2, ..., i\}$ and the vertex set of C_j by $V = \{1, 2, ..., j\}$. Notice that the numbering of the vertices for C_i and C_j must be consistent. In other words, the edge set of C_i is $\{(k, k+1) \mid k = 1, ..., i-1\} \cup \{(i, 1)\}$, and the edge set of C_j is $\{(k, k+1) \mid k = 1, ..., j-1\} \cup \{(j, 1)\}$.*

SOLUTION: **False , because $(i, 1)$ is an edge in C_i, but not an edge of C_j.**

Question 3: All of the following parts are based on the following undirected simple graph $G = (V, E)$.

$V = \{1, 2, ..., 8\}$, $E = \{(6, 5), (5, 3), (5, 4), (4, 3), (3, 1), (3, 2), (1, 2), (1, 8), (2, 8), (8, 7), (7, 6)\}$.

1. Is G bipartite? Explain.

   SOLUTION: **No, because if you color vertex 4 with blue and 3 with red, then 5 is adjacent to 4, so it must be red. But, on the other hand, 5 is also adjacent to 3, so it must be blue.**
2. Does G have Hamiltonian cycles? If yes, mention one (*only one*).

   SOLUTION: **Yes, consider $(4, 5, 6, 7, 8, 2, 1, 3, 4)$.**
3. Does G have Euler cycles? If yes, mention one. If not, explain why?

   SOLUTION: **No, because not all vertices are of even degree.**
4. Is **(3, 4, 5, 3, 1, 2, 3)** a cycle in G? Is it a simple cycle?

   SOLUTION: **It is a cycle, but not a simple sycle.**
5. Let H be the undirected simple graph with a vertex set $\{3, 4, 5, 2, 8\}$ and with an edge set $\{(3, 5), (3, 4), (2, 8)\}$.
   - Is H a subgraph of G? explain.

     SOLUTION: **Yes , because every vertex of H is a vertex of G and every edge of H is an edge of G.**
   - Is every simple path in $\overline{H}$ is also a simple path in $\overline{G}$? Explain.

     SOLUTION: **No, $(4, 5)$ is a path (*of length 1*) in $\overline{H}$ but not a path in $\overline{G}$.**
   - Is every simple cycle in $\overline{H}$ is also a simple cycle in $\overline{G}$? Explain.

     SOLUTION: **No, $(4, 5, 2, 4)$ is a simple cycle in $\overline{H}$, but not a simple cycle in $\overline{G}$.**
6. What is the degree sequence of G.

   SOLUTION: **4, 3, 3, 3, 3, 2, 2, 2.**
7. How many edges does $\overline{G}$ have? Explain.

   SOLUTION: $\frac{8 \cdot (8-1)}{2} - 11 = 17$**.**

8. Is there a path in G from vertex 1 to vertex 6 with no repeated edges and that includes all the edges of G and all the vertices of G? Explain.

   SOLUTION: **No, because 6 is of even degree. Notice also that we have 4 vertices of odd degree.**