Homework 2

Question 1: Let $A = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$, $B = \{(x, y) \in \mathbb{R}^2 \mid |x| \leq 1 \text{ and } |y| \leq 1\}$. Let $M = \mathbb{Z} \cup \{\frac{1}{2}\}$ and $L = \{x \mid x = \frac{1}{2} + q, q \in \mathbb{Z}\}$. Finally, let $C = A \cap (M \times M)$, $D = B \cap (M \times M)$, $E = A \cap (L \times L)$, $F = B \cap (L \times L)$. Find

(a) $D - C$.

(b) $F \cap D$.

Question 2:

(a) Give an example of a bijective function from $\mathbb{Z}$ onto $\mathbb{N}$.

(b) Show using the defintion of one-to-one that the function $f : \mathbb{N} \longrightarrow [e^{\frac{1}{4}}, e)$ defined by $f(x) = e^{\frac{x}{x+3}}$ is one-to-one.

Question 3: The following questions are unrelated.

(a) Let R be the relation on $\mathbb{Z}$ defined by

$$a \ R \ b \text{ if and only if } a = |b|^3.$$

Is R antisymmetric? Is is transitive? If the answer to any of them is yes, prove that. If the answer is not, then write down a counter example. Also, find R^{-1}.

(b) Let R be the relation on $\mathbb{Z}$ defined by

$$a \ R \ b \text{ if and only if } 3 - b \leq a - 2b < 1 + b.$$

Give a counter example to show that R is not transitive. Is R symmetric? If yes, prove it. If not write down a counterexample. Think (but do not hand in) about how to find R^{-1}.

(c) Let R be the relation on $\mathbb{Z}$ defined by

$$a \ R \ b \text{ if and only if } a - b \text{ is divisible by 5.}$$

This relation is an equivalence relation. Find $[17]$.

Question 4: The following parts are unrelated.

(a) Express the decimal number 578 in octal.

(b) Express the hexadecimal number 1E2B in decimal.

(c) Add the two binary numbers 10111 and 110011 using binary addition.

Question 5: Let the sequence a be defined as follows:

$$a_n = 1 + (-1)^n \text{ if } n \text{ is an odd natural number and } a_n = 2n - 1 \text{ if } n \text{ is an even natural number.}$$

Let $S_n = \sum_{k=1}^{n} a_k$.

(a) Find a general formula for S_n, $n \in \mathbb{N}$.

(b) Is $\{a_n\}_{n=1}^{\infty}$ increasing? Is it decreasing? Explain.