Question 1: Prove the following by mathematical induction:

(a) $(6)(7^n) - (2)(3^n)$ is divisible by 4, for every $n \in \mathbb{N}$.

Solution:

(1) *Basic Step:*

$(6)(7) - (2)(3) = 36$ is divisible by 4. Thus, the statement is true for $n = 1$.

(2) *Inductive Step:*

Assume that the statement is true for $n = k$, where $k \in \mathbb{N}$. In other words, assume that

$$(6)(7^k) - (2)(3^k) = 4m, \text{ for some } m \in \mathbb{Z}.$$

Notice that the above equation implies that $(6)(7^k) = 4m + (2)(3^k)$. Now depend on that to prove that the statement is true for $n = k + 1$. In other words, prove that

$$(6)(7^{k+1}) - (2)(3^{k+1}) = 4q, \text{ for some } q \in \mathbb{Z}.$$

But,

$$\begin{aligned}(6)(7^{k+1}) - (2)(3^{k+1}) &= (7)[(6)(7^k)] - (2)(3)(3^k) \\ &= 7[4m + (2)(3^k)] - (6)(3^k) \\ &= 28m + (14)(3^k) - (6)(3^k) \\ &= 28m + (14 - 6)(3^k) \\ &= 28m + (8)(3^k) \\ &= 4[7m + (2)(3^k)].\end{aligned}$$

Notice that $7m + (2)(3^k) \in \mathbb{Z}$. Hence, $(6)(7^{k+1}) - (2)(3^{k+1})$ is divisible by 4. Hence, the statement is true for $n = k + 1$.

Therefore, by mathematical induction, $(6)(7^n) - (2)(3^n)$ is divisible by 4, for every $n \in \mathbb{N}$.

(b) $(\frac{3}{2})^n \geq 1 + \frac{n}{2}$, for all $n \in \mathbb{N}$.

Solution:

(1) *Basic Step:*

$(\frac{3}{2})^1 \geq 1 + \frac{1}{2}$. Thus, the statement is true for $n = 1$.

(2) *Inductive Step:*

Assume that the statement is true for $n = k$, where $k \in \mathbb{N}$. In other words, assume that

$$(\tfrac{3}{2})^k \geq 1 + \tfrac{k}{2}$$

Now depend on that to prove that the statement is true for $n = k+1$. In other words, prove that

$$(\tfrac{3}{2})^{k+1} \geq 1 + \tfrac{k+1}{2} = \tfrac{3}{2} + \tfrac{k}{2}.$$

But,

$$\begin{aligned}(\frac{3}{2})^{k+1} &= (\frac{3}{2})(\frac{3}{2})^k \\ &\geq (\frac{3}{2})(1+\frac{k}{2}) \\ &= \frac{3}{2}+\frac{3}{4}k \\ &\geq \frac{3}{2}+\frac{k}{2}.\end{aligned}$$

Thus,

$$(\tfrac{3}{2})^{k+1} \geq 1+\tfrac{k+1}{2} = \tfrac{3}{2}+\tfrac{k}{2}.$$

Hence, the statement is true for $n = k+1$.

Therefore, by mathematical induction, $(\frac{3}{2})^n \geq 1+\frac{n}{2}$, for all $n \in \mathbb{N}$.

Question 2: Prove or disprove:

(a) $(p \longrightarrow q) \longrightarrow r \equiv p \longrightarrow (q \longrightarrow r)$.

(b) $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$.

Question 3: Prove or disprove:

(a) $x^4 - 4x^2 + 4$ is nonnegative for every real number x.

Solution:

The statement is true, because $x^4-4x^2+4 = (x^2-2)^2 \geq 0, \forall x \in \mathbb{R}$. Notice that the function $f(t) = t^2$ is nonnegative for every real number.

(b) $(n^2-5n+6)^3 + (1+(-1)^n)(2n+1)$ is an even integer for every natural number n.

Solution:

The statement is true. First, we have proved in class (you need to do it here) that n^2-5n+6 is even for all natural numbers. Thus, $n^2-5n+6 = 2q$, for some $q \in \mathbb{Z}$, and for all $n \in \mathbb{N}$. Now we have the following two cases:

Case 1: n is odd. In this case,

$$(n^2-5n+6)^3 + (1+(-1)^n)(2n+1) = (2q)^3 + (0)(2n+1) = 8q^3 = 2(4q^3).$$

Since $4q^3$ is an integer, it follows that $(n^2-5n+6)^3 + (1+(-1)^n)(2n+1)$ is even in this case.

Case 2: n is even. In this case,

$$(n^2-5n+6)^3 + (1+(-1)^n)(2n+1) = (2q)^3 + (2)(2n+1) = 2(4q^3+2n+1).$$

Since $4q^3+2n+1$ is an integer, it follows that $(n^2-5n+6)^3 + (1+(-1)^n)(2n+1)$ is even in this case also.

Since there are no other possibilities, the statement is true for all $n \in \mathbb{N}$.

(c) The sum of every two different prime numbers is an even integer.

Solution:

The statement is false. *Counterexample:* Take the two prime numbers 2 and 3. The sum of these two prime numbers is 5, which is not even.

(d) If a and b are irrational numbers, then a(b+1)+b is an irrational number.

Solution:

The statement is false. *Counterexample:* Take $a = \sqrt{2}$ and take $b = -a$. Then

$$a(b+1) + b = ab + a + b = ab = -(a^2) = -2$$

Thus, a and b are both irrational, but $a(b+1) + b$ is rational.

(e) The system

$$3x - y = 1$$
$$-6x + 2y = 7$$

has no real solution.

Solution: True. The proof is by contradiction. Assume the system has a real solution and try to slove it, you'll get something like $0 = 7$ which is absurd.

Question 4: Decide whether the following statements are true or false. Explain why.

(a) 8 is a prime number and 8 is an even integer $\longrightarrow$ either $\sqrt{5} = 1$ or $-2 > 0$.

Solution:

The statement is true. Notice that we have

$$(F \wedge T) \longrightarrow (F \vee F),$$

which simplifies to

$$F \longrightarrow F.$$

(b) If the only y-intercept of $x^2 + 4y^2 - 1 = 0$ is 1, then the only x-intercept of $e^x - 4y - 3 = 0$ is $ln(3)$.

Solution:

The statement is true. Notice that we have

$$F \longrightarrow T.$$