Definition 1. A *k-string Turing machine with input and output* is a k-string Turing machine with a read-only input string and write-only output string. The cursor of the first string does not move backward except when it encounters $\sqcup$. When it encounters $\sqcup$, it moves backward.

Definition 2. Suppose that for a $k-$string Turing machine M_k and an input x,

$$(s, \triangleright, x, \triangleright, \epsilon, ..., \triangleright, \epsilon) \xrightarrow{M^*} (H, w_1, u_1, ..., w_k, u_k),$$

where H is one of the halting states. Then the space required by M_k on x is $\Sigma_{i=1}^{k}|w_i u_i|$. If If M_k is with input and output, then the space required is: $\Sigma_{i=2}^{k-1}|w_i u_i|$.

Definition 3. Let $f : \mathbb{N} \longrightarrow \mathbb{N}$ and let M be a Turing machine, we say M operates within space bound $f(n)$ if the space required for any input x is less than or equal to $f(|x|)$.

Definition 4. SPACE($f(n)$): Languages decided by Turing machines with input and output that operate within space bound $f(n)$.

Definition 5. $L =$ SPACE(log$n$).

Definition 6. A *nondeterministic Turing machine* is a quadruple $N = (K, \Sigma, \triangle, s)$, where

$$\triangle \subset (K \times \Sigma) \times [(K \cup \{h, "yes", "no"\}) \times \Sigma \times \{\leftarrow, \rightarrow, -\}].$$

Notation: We will use **NDTM** for nondeterministic Turing machine and **DTM** for deterministic Turing machine.

Remark 0.1. Notice that if N is a NDTM, then for each combination of state and input symbol, there may be no action or more than one action.

Definition 7. NDTM's have applications in logic and AI.

Definition 8. We say

configuration (q, w, u) *yields* configuration (q', w', u') in *one step*,

denoted by

$$(q, w, u) \xrightarrow{N} (q', w', u')$$

if there exists a one-step transition of the machine from (q, w, u) to (q', w', u').
Similar definitions for

$$(q, w, u) \xrightarrow{N^k} (q', w', u'),$$

and

$$(q, w, u) \xrightarrow{N^*} (q', w', u')$$

Definition 9. Let N be a NDTM with alphabet Σ and let $x \in (\Sigma - \{\sqcup\})^*$. We say
(1) N accepts x if there is a sequence of choices that result in a "yes" output state.
(2) N rejects x if there is no sequence of choices leading to acceptance.

Definition 10. Let N be a NDTM and let Σ be an alphabet for M. Let L be a language over $\Sigma - \{\sqcup\}$ and let f be a function from the nonnegative integers to the nonnegative integers. We say that N decides L within times $f(n)$, if

(1) N decides L.

(2) $\forall x \in L$, if $(S, \triangleright, x) \xrightarrow{N^k} (q, w, u)$, then $k \leq f(|x|)$. In other words, N does not have computation paths greater than $f(n)$.

Remark 0.2. N decides L within space $f(n)$ is similar to the definition for M_k decides L within space $f(n)$, where M_k is a $k$string Turing machine with input and output.

Definition 11. **NTIME(f(n))**: Complexity class of languages that are decided by NDTM's operating within time $f(n)$.

Definition 12. P**:** set of all languages decidable by Turing machines in polynomial times (i.e. TIME(n^k), $k \geq 1$).

Definition 13. $NP = \text{NTIME}(n^k)$.

Theorem 14. *If L is decided by a NDTM N in time $f(n)$, then it is decided by a 3-string DTM in time $\mathcal{O}(c^{f(n)})$, where $c > 1$ is a constant depending on N.*

Remark 0.3. The above theorem says that any NDTM can be simulated by a DTM with exponential loss of efficiency.