Graphs

I. Definitions

Definition 1. A graph (undirected graph) G consists of a set of vertices and a set of edges. Each edge is associated with an unordered pair of vertices. If there is a unique edge e associated with the vertices u and v, then the edge e is expressed as (u, v) or (v, u). Edge e is said to be incident on u and v, and u and v are said to be incident on e. Also, u and v are said to be adjacent (neighbors). If the vertex set of G is V and the edge set is E, then we write $G = (V, E)$.

Definition 2. A directed graph (digraph) G consists of a set of vertices and a set of edges. Each edge is associated with an ordered pair of vertices. If there is a unique edge e from u to v, then the edge e is expressed as (u, v). Edge e is said to be incident on u and v, and u and v are said to be incident on e. Also, u and v are said to be adjacent (neighbors). If the vertex set of G is V and the edge set is E, then we write $G = (V, E)$.

Definition 3. Edges associated with the same pair of vertices are said to be parallel edges. An edge incident on a single vertex is called a loop.

Definition 4. A graph with no parallel edges and no loops is called a simple graph.

Definition 5. Let $G = (V, E)$ be a graph and let $v \in V$. The degree of v, denoted $\delta(v)$, is the number of edges incident on v. A vertex is called even if its degree is even and it is called odd if its degree is odd.

Definition 6. Let $G = (V, E)$ be a graph. The degree sequence of G is the sequence consisting of the degrees of all vertices of G ordered in a descending order.

Definition 7. Let $G = (V, E)$ be a graph and let v_0 and v_n be in V. A path from v_0 to v_n of length n is an alternating sequence of $n+1$ vertices and n edges beginning with v_0 and ending with v_n,

$$(v_0, e_1, v_1, e_2, v_2, ..., v_{n-1}, e_n, v_n),$$

in which edge e_i is incident on vertices v_{i-1} and v_i, $i = 1, ..., n$.

Definition 8. A graph $G = (V, E)$ is connected if for every u and v in V, there is a path in G between u and v. In other words, G is connected iff it has one piece (component) only.

Definition 9. Let $G = (V, E)$ be a simple undirected graph. The complement of G, denoted $\overline{G}$, is the simple undirected graph whose vertex set is V and whose edge set is

$$\overline{E} = \{(u, v) \mid u, v \in V,\ (u, v) \notin E\}.$$

Definition 10. Let $G = (V, E)$ and $H = (V', E')$ be two graphs. H is said to be a subgraph of G iff $V' \subseteq V$ and $E' \subseteq E$.

Definition 11. Let $G = (V, E)$ be a simple (undirected) graph and let $V' \subseteq V$. Then V' is said to be a clique iff for every u and v in V', $(u, v) \in E$.

Definition 12. Let $G = (V, E)$ be a simple (undirected) graph and let $V' \subseteq V$. Then V' is said to be an independent set iff for every u and v in V', $(u, v) \notin E$.

Definition 13. Let $G = (V, E)$ be a simple (undirected) graph and let $V' \subseteq V$. Then V' is said to be a vertex cover iff for every edge $e = (u, v) \in E$, either $u \in V'$ or $v \in V'$ (or both of them are in V').

Definition 14. Let $G = (V, E)$ be a graph and let $v \in V$. Then

- A simple path is a path with no repeated edges.
- A cycle (circuit) is a path from v to v with no repeated edges.
- A simple cycle is a cycle from v to v with no repeated vertices except v which must show up exactly twice (at the beginning and at the end of the loop).
- A Hamiltonian cycle is a simple cycle that includes all of the vertices of G.
- An Euler cycle is a cycle that includes all of the vertices and all of the edges of G.
- If there are numbers (weights) on the edges of G, then G is called a weighted graph.
- The length of a path in G is the sum of the weights of the edges in the path.

- a traveling salesman tour in a weighted graph is a Hamiltonian cycle of minimum length. In other words, it is the shortest tour that visists every vertex exactly once (except the starting and ending one) and ends the tour at the same vertex he started his tour at.

Definition 15. Let $G = (V, E)$ be an undirected graph. Then the adjacency matrix $A = (a_{ij})$ of G is the matrix in which a_{ij} is the number of edges from the vertex whose index is i to the vertex whose index is j.

II. Special Graphs

Definition 16. A complete graph on n vertices, denoted K_n, is a simple (undirected) graph in which every two (distinct) vertices are adjacent.

Definition 17. A null graph on n vertices, denoted N_n, is a simple (undirected) graph in which the edge set is empty.

Definition 18. Let $G = (V, E)$ be a simple (undirected) graph and let $V = \{1, 2, ..., n\}$ and

$$E = \{(i, i+1) \mid i = 1, ..., n-1\} \cup \{(n, 1)\}.$$

Then G is called a cyclic graph on n vertices.

Definition 19. Let $G = (V, E)$ be a simple (undirected) graph. Then G is called bipartite if V can be partitioned into two nonempty disjoint sets V_1 and V_2 such that $V = V_1 \cup V_2$, and such that no vertcies of V_1 are adjacent and no vertices of V_2 are adjacent. In other words, if $e \in E$, then one end of e must be in V_1 and the other must be in V_2.

Definition 20. The complete bipartite graph on m and n vertices, denoted $K_{m,n}$, is the simple undirected graph in which the vertex set V is partitioned into two nonempty disjoint sets V_1 and V_2 such that $V = V_1 \cup V_2$, $|V_1| = m$, $|V_2| = n$, and such that every vertex of V_1 is adjacent to every vertex of V_2 and no vertcies of V_1 are adjacent and no vertices of V_2 are adjacent.

Definition 21. Let $n \geq 2$. An $n - cube$ is a simple (undirected) graph with 2^n vertices labeled $0, ..., 2^n - 1$ and such that two vertices are adjacent iff the binary representations of these two vertices differ by one digit.

Definition 22. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two undirected graphs. Then G_1 and G_2 are said to be isomorphic if there is a bijective function f from V_1 onto V_2 such that

1. If $(u, w) \in E_1$, then $(f(u), f(w)) \in E_2$.
2. If $(y, z) \in E_2$, then there exists $(a, b) \in E_1$ such that $y = f(a)$, $z = f(b)$.

Notice that the second condition in the previous definition is necessary, because the first alone would mean that G_1 is isomorphic to a subgraph of G_2.

III. Theorems

Theorem 23. *Let $G = (V, E)$ be a graph and let $V = (v_1, v_2, ..., v_n)$. Then*

- $\sum_{i=1}^{n} \delta(v_i) = 2 \cdot |E|$. *Thus, the sum of the degree sequence is $2 \cdot |E|$.*
- *The number of odd vertices is even.*

Theorem 24. *A graph has an Euler cycle iff it is connected and all vertices are even.*

Theorem 25. *Let $G = (V, E)$ be a graph and let u and v be in V and $u \neq v$. Then there is a path in G between u and v with no repeated edges and that includes all vertcies and all edges of G iff G is connected and u and v are the only odd vertices of G.*

Theorem 26. *If a graph contains a cycle from v to v, then it contains a simple cycle from v to v.*

Facts

1. Let $G = (V, E)$ be a simple graph and let $u \in V$. If the degree of u in G is r, then the degree of u in $\overline{G}$ is $(n - 1) - r$.

2. Let $G = (V, E)$ be a simple graph and let $n = |v|$. If the degree sequence of G is $a_1, a_2, ..., a_n$, then the degree of $\overline{G}$ is $(n-1) - a_n, (n-1) - a_{n-1}, ..., (n-1) - a_2, (n-1) - a_1$.
3. If the degree sequence of a simple graph G is $a_1, a_2, ..., a_n$, then the number of edges of G is equal to the sum of the degree sequence divided by 2.
4. Let $G = (V, E)$ be a simple graph. Then the space complexity of the adjacency matrix representation is $\mathcal{O}(|V|^2)$.
5. Let $G = (V, E)$ be a simple graph. Then the adjacency matrix representation is used when G is dense. I.e. when $|E| \approx |V|^2$.
6. Let $G = (V, E)$ be a simple graph. Then the space complexity of the adjacency list representation is $\mathcal{O}(|E| + |V|)$.
7. Let $G = (V, E)$ be a simple graph. Then the adjacency list representation is used when G is sparse. I.e. when $|E| << |V|^2$.

Facts Related to Adjacency Matrices

Let $G = (V, E)$ be an undirected graph whose adjacency matrix $A = (a_{ij})$. Notice that a_{ij} is the number of edges from the vertex whose index is i to the vertex whose index is j. Notice also that A is symmetric. Then the degree of the vertex whose index is i is equal to the sum of the elements in row (or column) i of A.

If the graph in addition is simple, then

1. The elements of A should be either 0 or 1 and the main diagonal must be zero.
2. If the adjacency matrix of $\overline{G}$ can be obtained from A by changing the zero elements of A to ones (except those on the main diagonal which should remain zero) and the ones elements of A to zeros.
3. The ith element of the diagonal of A^2 is equal to the degree of the vertex whose index is i.
4. The previous step enables you to find the degree sequence of A, the degree sequence of $\overline{G}$, the number of edges of G and the number of edges of $\overline{G}$. To get the degree sequence of G, all you need to do is to find the degree of each vertex as described above and then order them in a decreasing order. That is, the

degree sequence of G is simply the elements of the diagonal of A^2 ordered in a decreasing order. To get the degree sequence of $\overline{G}$, subtract each element of the degree sequence of G from $n-1$, where $n = |V| =$ number of rows of A^2. Then order the new elements you'll get in a decreasing order. Now to find the number of edges of G, simply find the sum of the degree sequence of G and then divide the result by 2. Finally, to get the number of edges of $\overline{G}$, simply subtract the number of edges of G from $\frac{n(n-1)}{2}$. each element of the

5. If you know the adjacency matrix of G, then you can esaily know the adjacency matrix of $\overline{G}$. How? And, hence, you can know the graph of $\overline{G}$.
6. If you know the adjacency matrix of G, then you can esaily know the graph of G.
7. If we denote A^m (m here is a natural number) by $B = (b_{ij})$, then b_{ij} represents the number of paths of length m from the vertex whose index is i to the vertex whose index is j.
8. If you know the number of paths of length m from the vertex whose index is i to the vertex whose index is j, then you know the number of paths of length $k \cdot m$ ($k \in \mathbb{N}$) from the vertex whose index is i to the vertex whose index is j. How? Answer: simply compute A^{km} which is equal to $(A^m)^k$.
9. If G is weighted, then the ijth element of A is replaced by the weight of the edge connceting the vertex whose index is i and the vertex whose index is j.

Facts About Isomorphic Graphs

If $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are isomorphic, then there structure should be the same, because isomorphisms preserve structure. Thus, if G_1 and G_2 are isomorphic, then

1. The degree sequence of G_1 must be the same as the degree sequence of G_2. For instance, if G_1 has no vertex of degree 3, then G_2 must have no vertex of degree 3, and vice versa. If G_1 has a vertex of degree 3, then G_2 must have a vertex of degree 3, and vice versa. If G_1 has exactly 4 vertices of degree 3, then G_2 must have exactly 4 vertices of degree 3, and vice versa.

Notice that the above also means that the number of vertices of G_1 must be the same as the number of vertices of G_2 and the number of edges of G_1 must be the same as the number of edges of G_2. In general, if G_1 has k vertices of degree m, then G_2 must have k vertices of degree m and vice versa.

2. The number of pieces of G_1 must be the same as the number of pieces of G_2.
3. The number and kind of cycles of paths G_1 must be the same as the number and kind of cycles of paths G_2.
4. If G_1, has, say, 2 adjacent vertices, say, of degrees 2 and 6, then G_2 must have two adjacent vertices of degrees 2 and 6. In general, if G_1 has k vertices of degree d_1, d_2, ..., d_k that are adjacent, then G_2 must have the same thing and vice versa.
5. If G_1 is, say, bipartite, then G_2 must be bipartite and vice versa.

Remark: Notice that if all the conditions above are satisfied, then that does not necessarily mean that G_1 and G_2 are isomorphic. On the other hand, if at least one of them is not satisfied, then they cannot be isomorphic.