

Functions

DEFINITION 1. A *function* f is a relation from a set A to a set B such that every element in A is related to exactly one element in B . A is called the *domain* of f and B is called the *range* of f . We write $f : A \longrightarrow B$.

The definition of a function is usually stated as follows: $f : A \longrightarrow B$ is a function if every element in A has an image in B and whenever $x_1 = x_2$, we must have $f(x_1) = f(x_2)$.

EXAMPLE 2. Let $A = \{a, b, c\}$, let $B = \{1, 2, 3\}$, and let $R_1 = \{(a, 1), (b, 2), (c, 2)\}$, $R_2 = \{(a, 1), (b, 2)\}$, and $R_3 = \{(a, 1), (a, 2), (b, 2), (c, 3)\}$. Then, R_1 defines a function, but R_2 and R_3 do not. R_2 is not a function, because c has no image in B , and R_3 is not a function, because a has two different images.

DEFINITION 3. A function $f : A \longrightarrow B$ is said to be *one-to-one* (or *injective*) if whenever $f(x_1) = f(x_2)$, we must have $x_1 = x_2$.

In other words, f is one-to-one if no two points in the domain share the same image.

DEFINITION 4. A function $f : A \longrightarrow B$ is said to be *onto* (or *surjective*) if for every $b \in B$, there exists $a \in A$, such that $f(a) = b$.

In other words, f is onto if every element in B has a preimage from A .

DEFINITION 5. A function $f : A \longrightarrow B$ is said to be *bijective* if it is one-to-one and onto.

DEFINITION 6. If $f : A \longrightarrow B$ is bijective, then A and B are said to have the same *cardinality*. If this is the case, then we write $|A| = |B|$.

DEFINITION 7. Let f be a function from A to B . If f is a bijection, then the inverse of f , denoted by f^{-1} , is defined to be:

$$f^{-1} = \{(y, x) \mid (x, y) \in f\}.$$

Notice that if f is a function from A to B and if f^{-1} is defined, then f^{-1} is a function from B to A .

EXAMPLE 8. Consider $f : \mathbb{R} \longrightarrow \mathbb{R}^+$, defined by $f(x) = e^x, \forall x \in \mathbb{R}$. Then, $f^{-1}(x) = \ln(x)$. Notice that the domain of f^{-1} is $\mathbb{R}^+$ and the range is $\mathbb{R}$.

EXAMPLE 9. Consider $f : \mathbb{R} \longrightarrow \mathbb{R}^+$, defined by $f(x) = 10^x, \forall x \in \mathbb{R}$. Then, $f^{-1}(x) = \log(x)$. Notice that the domain of f^{-1} is $\mathbb{R}^+$ and the range is $\mathbb{R}$.

EXAMPLE 10. Let $f : \mathbb{R} - \{\frac{1}{2}\} \longrightarrow \mathbb{R} - \{\frac{3}{2}\}$, be defined by:

$$f(x) = \frac{3x}{2x-1}, \forall x \in \mathbb{R} - \{\frac{1}{2}\}.$$

Then, f^{-1} is a function from $\mathbb{R} - \{\frac{3}{2}\}$ onto $\mathbb{R} - \{\frac{1}{2}\}$, and it's given by:

$$f^{-1}(x) = \frac{x}{2x-3}, \forall x \in \mathbb{R} - \{\frac{3}{2}\}.$$

Fact: If $f : A \longrightarrow B$ is bijective, then $f^{-1} : B \longrightarrow A$ is defined and it is also bijective.

DEFINITION 11. A set A is said to be *countably infinite* if there is a bijection (a bijective function) from A to $\mathbb{N}$ or from $\mathbb{N}$ to A . In other words, A is countable infinite if $|A| = |\mathbb{N}|$. A set A is said to be *countable* if it is either *finite* or *countably infinite*.

EXAMPLE 12. Let $A = \{a, b, c\}$, $B = \{1, 2, 3\}$, and define the following functions from A to B .

- $f(a) = 1, f(b) = 1, f(c) = 2$. Then, f is not one-to-one, because a and b share the same image. Notice that $f(a) = f(b)$. f is not onto, because there is no element in A whose image is 3.
- $f(a) = 1, f(b) = 2, f(c) = 3$. Then, f is one-to-one and onto.

EXAMPLE 13. The function $f : \mathbb{R} \longrightarrow \mathbb{R}$, defined by $f(x) = x^2 - 5x + 6$, is not one-to-one and not onto. Not one-to-one, because, for example, $f(1) = f(4)$. One way to find such points is the following: assume $f(x_1) = f(x_2)$, and then solve the resulting

equation, to get: $x_1^2 - 5x_1 + 6 = x_2^2 - 5x_2 + 6$. This implies that $x_1^2 - x_2^2 = 5(x_1 - x_2)$. Thus, $(x_1 - x_2)(x_1 + x_2) = 5(x_1 - x_2)$. Now we have two possibilities: either $x_1 - x_2 = 0$, or $x_1 - x_2 \neq 0$. If $x_1 - x_2 \neq 0$, then we can divide both sides by this quantity, to get $(x_1 + x_2) = 5$.

Now f is not onto, because, for example, there is no $a \in \mathbb{R}$ such that $f(a) = -1$. To prove it, assume that such a exists, then $a^2 - 5a + 6 = -1$. Thus, $a^2 - 5a + 7 = 0$. Try to solve this equation, you get two complex solutions. Thus, since it has no real solutions, then there is no $a \in \mathbb{R}$ such that $f(a) = -1$.

EXAMPLE 14. The function $f : \mathbb{R} \longrightarrow [-\frac{1}{4}, \infty)$, defined by $f(x) = x^2 - 5x + 6$, is onto but not one-to-one. Not one-to-one, because, for example, $f(1) = f(4)$ (the same reason as the previous example).

Notice that the lowest point on the graph of $f(x)$ is $(\frac{5}{2}, -\frac{1}{4})$. In other words, the graph of f is entirely on or above the line $y = -\frac{1}{4}$. Thus, any horizontal line $y = c$, where $c \geq -\frac{1}{4}$, intersects the graph of f at least once and so f has to be onto. It's easy to prove "formally" that f is onto, because if $b \in B$, where $B = [-\frac{1}{4}, \infty)$ then the equation has $a^2 - 5a + 6 = b$ has a solution from the domain, $\mathbb{R}$. Notice here that you get $a = \frac{5 \pm \sqrt{1+4b}}{2}$. Now note that since $b \in B$, then $b \geq -\frac{1}{4}$. Thus, $1 + 4b \geq 0$. Therefore, the above equation always has a real solution, say that a_1 is one of them. Then, it follows that $f(a_1) = b$.

EXAMPLE 15. The function $f : (-\infty, \frac{5}{2}] \longrightarrow [-\frac{1}{4}, \infty)$, defined by $f(x) = x^2 - 5x + 6$, is onto and one-to-one. $f(x)$ is onto for the same explanation offered in the previous example. Now f is one-to-one, because if $f(x_1) = f(x_2)$, where x_1 and x_2 are in $A = (-\infty, \frac{5}{2}]$, then $x_1^2 - 5x_1 + 6 = x_2^2 - 5x_2 + 6$, which implies that $(x_1 - x_2)^2 = 5(x_1 - x_2)$. Now we have two cases:

- **Case 1:** $x_1 - x_2 \neq 0$. In this case, we get $x_1 + x_2 = 5$. Notice now that either $x_1 = x_2 = \frac{5}{2}$ or one of them (I mean x_1, x_2) has to be greater than $\frac{5}{2}$, which makes it not in the domain A . So, no two different elements of A can share the same value of f in this case.
- **Case 2:** $x_1 - x_2 = 0$. In this case, we get $x_1 = x_2$.

So, in all cases, no two different elements of A can share the same value of f .

Now notice that since f is one-to-one and onto, then it is bijective, and hence, $(-\infty, \frac{5}{2}]$ and $[\frac{-1}{4}, \infty)$ have the same cardinality.

EXAMPLE 16. The function $f : \mathbb{Z} \longrightarrow \mathbb{Z}$, defined by $f(x) = x^2 - 5x + 6$, is not one-to-one and not onto. Not one-to-one because of the same reasons stated earlier in one of the examples related to this function. Remember that you need to set $f(x_1) = f(x_2)$ and then solve the resulting equation. Do so, you'll get $(x_1 - x_2)^2 = 5(x_1 - x_2)$. Thus, either $x_1 = x_2$ or $x_1 + x_2 = 5$. But, the equation $x_1 + x_2 = 5$ has solutions from $\mathbb{Z}$. For example, $x_1 = 1$ and $x_2 = 4$ are solutions. This implies $f(1) = f(4)$.

Now f is not onto, because there is no $a \in \mathbb{Z}$ such that $f(a) = 1$. If such a exists, then we must have $a^2 - 5a + 6 = 1$, which implies $a = \frac{5 \pm \sqrt{5}}{2}$. The two values of a which we get are not integers, because $\sqrt{5}$ is irrational.

EXAMPLE 17. The function $f : \mathbb{Z} \longrightarrow \mathbb{N} \cup \{0\}$, defined by $f(x) = x^2 - 5x + 6$, is not one-to-one and not onto. Not one-to-one and not onto because of the same reasons stated in the previous example.

EXAMPLE 18. We know that $f(x) = e^x$ is a bijective function from $\mathbb{R}$ onto $(0, \infty)$. Try to prove that. This means that $\mathbb{R}$ and $(0, \infty)$ have the same cardinality. Now the question is: how to get a bijective function from $\mathbb{R}$ onto (b, ∞) ? The answer is by taking $f(x) = b + e^x$. The next question is: how to get a bijective function from $\mathbb{R}$ onto $(-\infty, b)$? The answer is by taking $f(x) = b - e^x$.

EXAMPLE 19. Let $f : \mathbb{N} \longrightarrow \mathbb{Z}$ be defined by $f(x) = \frac{x}{2}$ if x is even, and $f(x) = \frac{1-x}{2}$ if x is odd. Then f is bijective (Prove it). Hence, $\mathbb{Z}$ and $\mathbb{N}$ have the same cardinality. Therefore, $\mathbb{Z}$ is countably infinite. Now notice that f^{-1} is defined and it is a bijective function from $\mathbb{Z}$ onto $\mathbb{N}$. Try to find f^{-1} . Does this remind you of anything related to the homework?

Notation: We will use the set $2\mathbb{N} - 1$ to denote the set of *odd* natural numbers (i.e. the set $\{2n - 1 \mid n \in \mathbb{N}\}$). And we will use the set $2\mathbb{N}$ to denote the set of *even* natural numbers (i.e. the set $\{2n \mid n \in \mathbb{N}\}$).

EXAMPLE 20. The function $f : \mathbb{N} \longrightarrow 2\mathbb{N} - 1$, defined by $f(n) = 2n - 1$ is bijective and so $2\mathbb{N} - 1$ is countably infinite. Thus, $|2\mathbb{N} - 1| = |\mathbb{N}|$.

EXAMPLE 21. The function $f : \mathbb{N} \longrightarrow 2\mathbb{N}$, defined by $f(n) = 2n$ is bijective and so $2\mathbb{N}$ is countably infinite. Thus, $|2\mathbb{N}| = |\mathbb{N}|$.

EXAMPLE 22. The function $f : \mathbb{N} - \{1, 2, 3, 4\} \longrightarrow \mathbb{N}$, defined by $f(n) = n - 4$ is bijective and so $\mathbb{N} - \{1, 2, 3, 4\}$ is countably infinite. Thus, $|\mathbb{N} - \{1, 2, 3, 4\}| = |\mathbb{N}|$.

EXAMPLE 23. The function $f : \mathbb{N} - \{1, 2, 3, 4\} \longrightarrow 2\mathbb{N} - 1$, defined by $f(n) = 2(n - 4) - 1 = 2n - 9$ is bijective and so $|\mathbb{N} - \{1, 2, 3, 4\}| = |2\mathbb{N} - 1| = |\mathbb{N}|$.

EXAMPLE 24. The function $f : \mathbb{N} - \{1, 2, 3, 4\} \longrightarrow 2\mathbb{N}$, defined by $f(n) = 2(n - 4) = 2n - 8$ is bijective and so $|\mathbb{N} - \{1, 2, 3, 4\}| = |2\mathbb{N}| = |\mathbb{N}|$.

EXAMPLE 25. In this example, we will find a bijective function from the interval (a, b) onto the interval (c, d) , where none of a , b , c , and d is equal to $\pm\infty$. The function is given by:

$$f(x) = c + \frac{d-c}{b-a}(x-a), \forall x \in (a, b).$$

Notice that f is not the straight line $y = c + \frac{d-c}{b-a}(x-a)$. f is the segment of the previously- mentioned straight line which lies between $x = a$ and $x = b$.

EXAMPLE 26. Is it possible to find a bijective function from the interval $(1, 2)$ onto the interval $(3, 7)$? If yes, give an example.

Solution:

Yes. The function $f : (1, 2) \longrightarrow (3, 7)$, defined by $f(x) = 4x - 1$, is bijective.

EXAMPLE 27. Is it possible to find a bijective function from the interval $[1, 2)$ onto the interval $[3, 7)$? If yes, give an example.

Solution:

Yes. The function $f : [1, 2) \longrightarrow [3, 7)$, defined by $f(x) = 4x - 1$, is bijective.

EXAMPLE 28. Is it possible to find a bijective function from the interval $[1, 2)$ onto the interval $(3, 7]$? If yes, give an example.

Solution:

Yes. The function $f : [1, 2) \longrightarrow (3, 7]$, defined by $f(x) = 4x - 1$, $\forall x \in (1, 2)$, and $f(1) = 7$, is well-defined and bijective.

Question: Is it possible to find a bijective function from the interval $(1, 2)$ onto the interval $[3, 7]$? If yes, give an example.

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EXAMPLE 29. Prove that the function $f : (\mathbb{N} \cup \{0\}) \times (\mathbb{N} \cup \{0\}) \longrightarrow \mathbb{N} \cup \{0\}$, defined by $f(k, n) = 2^k(2n + 1) - 1$ is one-to-one.

Solution: Assume $f(k_1, n_1) = f(k_2, n_2)$ for some (k_1, n_1) and (k_2, n_2) in $(\mathbb{N} \cup \{0\}) \times (\mathbb{N} \cup \{0\})$. Our goal is to show that $(k_1, n_1) = (k_2, n_2)$. Notice that $(k_1, n_1) = (k_2, n_2)$ iff $k_1 = k_2$ and $n_1 = n_2$. But, since $f(k_1, n_1) = f(k_2, n_2)$, we have that $2^{k_1}(2n_1 + 1) - 1 = 2^{k_2}(2n_2 + 1) - 1$. Now without loss of generality, assume that $k_1 \geq k_2$, and then rewrite the last equation as: $2^{k_1 - k_2} = \frac{2n_2 + 1}{2n_1 + 1}$. Now notice that the right hand side is either a noninteger fraction or it is odd. On the other hand, the left hand side is either 1 or a multiple of 2. (Notice that $k_1 \geq k_2$ by assumption.) The only way this can happen is if both sides are equal to 1. This yields to the desired result.

DEFINITION 30. Let f be a function from X to Y and g a function from Y to Z . Then the composition $g \circ f$ is defined as follows: $g \circ f(x) = g(f(x))$, for all x in the domain of $g \circ f$.

Notice that in order for x to be in the domain of $g \circ f$, x has to be in the domain of f and $f(x)$ has to be in the domain of g . Notice also that if $f : A \longrightarrow B$ is invertible; i.e. f^{-1} exists, then $f \circ f^{-1}(x) = x$, $\forall x \in B$ and $f^{-1} \circ f(x) = x$, $\forall x \in A$.

EXAMPLE 31. Show that the cardinality of $\mathbb{R}^+$ is equal to the cardinality of the interval $(0, 1)$ by presenting a bijective function from $\mathbb{R}^+$ onto $(0, 1)$. Then depend on that function to find a bijective function from $(0, 1)$ onto $\mathbb{R}$. Finally, prove that the cardinality of $\mathbb{R}$ is equal to the cardinality of the interval $(0, 1)$.

Solution:

Consider the function $f : \mathbb{R}^+ \longrightarrow (0, 1)$, defined by $f(x) = \frac{x}{x+1}$, $\forall x \in \mathbb{R}^+$. It is easy to prove that this function is bijective. (The proof is left as an exercise.) Now consider the function $g : \mathbb{R} \longrightarrow \mathbb{R}^+$, defined by $g(x) = e^x$, $\forall x \in \mathbb{R}$. It is easy to prove that this function is bijective. Finally, consider the function $h : \mathbb{R} \longrightarrow (0, 1)$, defined by $h(x) = f \circ g(x) = \frac{e^x}{e^x+1}$, $\forall x \in \mathbb{R}$. This function is bijective (why?).

Questions on Functions

- Define a bijective function from $\mathbb{N} \cup \{-2, -1, 0\}$ onto $2\mathbb{N} - 1$.
- Define a bijective function from $\mathbb{N} \cup \{-2, -1, 0\}$ onto $A - \{1, 3, 5, 7\}$, where $A = 2\mathbb{N} - 1$.
- Let $f(x)$ be defined as follows:

$$f(x) = \frac{4x-1}{3x-7}.$$

Find the largest possible domain of f and find the range of f corresponding to the domain you've found. Call the domain A . Now find the inverse of f defined (I mean f) over A . What are the domain and range of f^{-1} ?

- Let $f(x)$ be defined as follows:

$$f(x) = \ln(3x - 8).$$

Find the largest possible domain of f and find the range of f corresponding to the domain you've found. Call the domain A . Now find the inverse of f defined (I mean f) over A . What are the domain and range of f^{-1} ?

- Let A be the interval $(4, 7)$ and let B be the interval $(2, 3)$. Find a bijective function from A onto B .
- Let A be the interval $(4, \infty)$ and let B be the interval $(2, 3)$. Find a bijective function from A onto B .

- Let A be the set of integers and let B be the set of all equivalence classes corresponding to the relation R on $\mathbb{Z}$ defined as follows

$$aRb \text{ iff } a - b \text{ is a multiple of } 5$$

Now define the function f from A to B as follows:

$$f(x) = [x], \forall x \in \mathbb{Z},$$

where $[x]$ is the equivalence class of x (as it's defined by the relation R .)

Is f one-to-one? Is it onto? Explain.

- Prove or disprove: If $f : A \longrightarrow B$ and $g : B \longrightarrow C$ are one-to-one and onto, then so is $g \circ f$.
- Is it possible to find a bijective function from the interval $(0, 1)$ onto the interval $(0, 2)$? If yes, give an example.
- Is it possible to find a bijective function from the interval $(0, 1)$ onto the interval $[0, 2]$? If yes, give an example.
- Is it possible to find a bijective function from the interval $(0, 1)$ onto the interval $[0, 2]$? If yes, give an example.
- Is it possible to find a bijective function from $\mathbb{R}^+$ onto the interval $(0, \frac{1}{3})$? If yes, give an example.
- Is it possible to find a bijective function from $\mathbb{R}$ onto the interval $(0, \frac{1}{3})$? If yes, give an example.
- Is it possible to find a bijective function from $(\mathbb{N} \cup \{0\}) \times (\mathbb{N} \cup \{0\})$ onto $\mathbb{Z}$? If yes, give an example.

More material will be added to this section in the near future.