Exam III Solutions

Question 1: Do only 3 of the following parts:

(a) (7 points) Solve $4x \equiv 4 \pmod{6}$.

Solution: $x \equiv 1$ **(mod 6),** $x \equiv 4$ **(mod 6).**

(b) (7 points) Solve $4x \equiv 8 \pmod{80}$.

Solution: $x \equiv 2$ **(mod 80),** $x \equiv 22$ **(mod 80),** $x \equiv 42$ **(mod 80),** $x \equiv 62$ **(mod 80).**

(c) (7 points) Solve $(4x + 4)(x + 3) \equiv 0 \pmod{6}$.

Solution: $x \equiv 0$ **(mod 6),** $x \equiv 2$ **(mod 6),** $x \equiv 3$ **(mod 6),** $x \equiv 5$ **(mod 6).**

(d) (7 points) Solve the system:

$x + y \equiv 1 \pmod{4}$

$x + 3y \equiv 1 \pmod{4}$.

Solution: $x \equiv 1$ **(mod 4),** $y \equiv 0$ **(mod 4), and** $x \equiv 3$ **(mod 4),** $y \equiv 2$ **(mod 4).**

Question 2: Do only 3 of the following parts:

(a) (7 points) Let $a = -36$ and $b = 15$. Find the quotient and the remainder when a is divided by b.

Solution: $q = -3$**,** $r = 9$**.**

(b) (7 points) Find gcd(36,25) and write it as a linear combination of 36 and 25.

Solution: **gcd(36,25)=1. Linear Combination:** $1 = (-9)(36) + (13)(25)$**.**

(c) (7 points) Find the multiplicative inverse of 25 (mod 36).

Solution: **From the previous part:** $1 = (-9)(36) + (13)(25)$**. Thus, since** $36 \equiv 0$ **(mod 36), it follows that the multiplicative inverse of 25 (mod 36) is 13.**

(d) (7 points) Find 36 (mod -15). Also, determine whether n and $3n$ are relatively prime or not, where n is a natural number greater than 1.

Solution: **36 (mod -15)=6. gcd(n,3n)=n. Since** $n \geq 2$**, then** n **and** $3n$ **are not relatively prime.**

Question 3: Do only 4 of the following parts. All graphs are simple and undirected.

(a) (8 points) Let G be a simple undirected graph. If the degree sequence of $\overline{G}$ is

4, 2, 2, 1, 1, 1, 1. What is the degree sequence of G?

Solution: **5, 5, 5, 5, 4, 4, 2.**

(b) (8 points) When does the $n-cube$ have an Euler cucle? When does C_n have an Euler cycle if $n \geq 7$?

Solution: **The** $n - cube$ **has an Euler cycle when** n **is even.**

C_n**,** $n \geq 7$**, has an Euler cycle always.**

(c) (8 points) Let $G = (V, E)$ be a simple undirected graph. Prove that if G is isomorphic to $\overline{G}$, then either $|V|$ or $|V| - 1$ is a multiple of 4.

Solution: **Done in class.**

(d) (8 points) How many edges does the $n-cube$ have? How many edges does $\overline{K_{19,35}}$ have?

Solution: **Number of edges of the** $n - cube$ **is** $n \cdot 2^{n-1}$**.**

Number of edges of $\overline{K_{19,35}}$ **is** $\frac{(19+35)(19+35-1)}{2} - (19)(35)$.

(e) (8 points) Let $G = (V, E)$ be the simple undirected graph defined by $V = \{v_1, v_2, v_3, ..., v_n\}$, $n \geq 100$, and

$$E = \{(v_i, v_j) \mid |i - j| = 1\}.$$

How many edges does G have? What is the degree sequence of G? How many edges does $\overline{G}$ have? Does G have a path from v_1 to v_n with no repeated edges and that includes all edges and all vertices of G? Explain.

Solution: The number of edges of G is $n - 1$. The number of edges of $\overline{G}$ is $\frac{(n-1)(n-2)}{2}$. Yes, there the described path exists because G is connected and 1 and n are the only vertices of odd degree.

(f) (8 points) Let G be a simple undirected graph with an adjacency matrix A. If the diagonal elements of A^2 are (not necessarily in order) 5, 1, 1, 2, 3, 4, 1, 1 , what are the diagonal elements of B^2, where B is the adjacency matrix of $\overline{G}$?

Solution: 2, 6, 6, 5, 4, 3, 6, 6.

(g) (8 points) Let G be a simple undirected graph with an adjacency matrix A, where

$$A = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \end{bmatrix}$$

Without graphing, determine the degree sequence of G and the adjacency matrix of $\overline{G}$.

Solution: Degree sequence: 4, 4, 3, 3, 2.

The adjacency matrix of $\overline{G}$ is:

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Question 4: Let $G = (V_1, E_1)$ be defined by $V_1 = \{1, 2, ..., 7\}$,

$$E_1 = \{(1,2), (1,7), (2,3), (2,7), (2,5), (3,4), (3,5), (4,5), (5,6), (6,7)\}.$$

Let $H = (V_2, E_2)$ be defined by $V_2 = \{2, 3, 5, 6\}$, $E_2 = \{(2,3), (2,5)\}$. Let $M = (V_3, E_3)$ be defined by $V_3 = \{1, 2, ..., 7\}$,

$$E_3 = \{(1,2), (1,7), (1,6), (2,7), (2,3), (2,5), (3,4), (4,6), (5,6), (6,7)\}.$$

(a) (5 points) Is H a subgraph of G?

Solution: Yes.

(b) (5 points) What is the degree sequence of G?

Solution: 4, 4, 3, 3, 2, 2, 2.

(c) (6 points) Is G isomorphic to M? Explain.

Solution: No, because in M 3 and 4 are of degree 2 and adjacent, while in G no vertices of degree 2 are adjacent.

(d) (5 points) Is G bipartite? Explain.

Solution: No. Use coloring. Notice that G contains a cycle of order 3. So, coloring fails on this cycle.

(e) (5 points) What is the adjacency matrix of G?

Solution:

The adjacency matrix of G is:

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$