Solutions of Exam I

Question 1:

(a)

a	b	$\overline{a}$	$a \longrightarrow b$	$\overline{a} \vee b$
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

(b)

(i) False. *Counterexample:* 2 is prime but not odd.

(ii) True. The proof is by cases:

If n is even, then $(-1)^n = 1$. Hence, $3 + (-1)^n = 4$ in this case. Therefore, it is even.

If n is odd, then $(-1)^n = -1$. Hence, $3 + (-1)^n = 2$ in this case. Therefore, it is even.

Question 2:

(a) $\{\{1, 2\}\}$. Be careful, $\{1, 2\}$ is a wrong answer.

(b)

(i) False.

(ii) False.

(iii) True.

Question 3:

(a) $\{1\}$.

(b) False. Let $U = \{1,2,3\}$, $A = \{1\}$, and $B = \{1,2\}$. Then, $\overline{A} = \{2,3\}$ and $\overline{B} = \{3\}$. Thus, $A \subseteq B$, but $\overline{A}$ is not a subset of $\overline{B}$.

Question 4:

(a) Not increasing. Consider a_2 and a_3. The index of a_2 is less than that of a_3, but $a_2 > a_3$.

Not decreasing. Consider a_1 and a_2. The index of a_1 is less than that of a_2, but $a_1 < a_2$.

(b) $\prod_{i \in S} b_i = b_4 * b_7 = 2^1 * 2^4 = 32$.

Question 5:

(a)

(i) Not antisymmetric, because $(1,2)$ and $(2,1)$ are both in R.

(ii) Not symmetric, because $(1,10) \in \mathbb{R}$, but $(10,1)$ is not.

(b)

$$\begin{aligned}
[3] &= \{a \in \mathbb{Z} - \{0\} \mid aR3\} \\
&= \{a \in \mathbb{Z} - \{0\} \mid 3a > 0\} \\
&= \{a \in \mathbb{Z} - \{0\} \mid a > 0\} \\
&= \mathbb{Z}^+.
\end{aligned}$$

R has two equivalence classes. We've already found one of them. Take any negative integer, e.g. -1, you get the other, which is: $\{a \in \mathbb{Z} - \{0\} \mid a < 0\} = \mathbb{Z}^-$.

Question 6:

(a) Yes, R is transitive. To prove it, let (a, b) and (b, c) be both in R, where a, b, and c are all in $\mathbb{Z}$. Now since $(a, b) \in R$, then there exists an integer m such that $3a+b = 4m$. And since $(b, c) \in R$, then there exists an integer k such that $3b+c = 4k$. Now, we have to show that $(a, c) \in R$. But, $3a+c = (4m-b)+(4k-3b) = 4(m+k-b)$. Thus, $(a, c) \in R$.

(b)

$$\begin{aligned}
[(1,3)] &= \{(x,y) \in \mathbb{R}^2 \mid (x,y)R(1,3)\} \\
&= \{(x,y) \in \mathbb{R}^2 \mid x + 2y = 1 + 6\} \\
&= \{(x,y) \in \mathbb{R}^2 \mid x + 2y = 7\}
\end{aligned}$$

Thus, $[(1, 3)]$ is the straight line in the cartesian plane that has a slope of $\frac{-1}{2}$ and a y-intercept of $\frac{7}{2}$.

Similarly, $[(1, 2)]$ is the straight line in the cartesian plane that has a slope of $\frac{-1}{2}$ and a y-intercept of $\frac{5}{2}$.

Question 7: As I've mentioned in the hint, it's sufficient to prove that

$$2 + 4 + 6 + ... + 2n = n(n + 1), \forall n \in \mathbb{N}.$$

Basic Step: For $n = 1$:

$1(1 + 1) = 2$. So, it's true for $n = 1$.

Inductive Step: Assume it's true for $n = k$, where $k \in \mathbb{N}$, and show that it's true for $n = k + 1$.

From the assumption, we have

$$2 + 4 + 6 + ... + 2k = k(k + 1).$$

Now

$$\begin{aligned} 2+4+6+...+2k+2(k+1) &= k(k+1)+2(k+1) \\ &= (k+1)(k+2) \end{aligned}$$

Thus, $2+4+6+...+2k+2(k+1) = (k+1)(k+2)$. Therefore, the statement is true for $n = k+1$. Hence, by the Principle of Mathematical Induction, the statement is true.

Question 8:

(a) The answer is any member (except ϕ) of the power set of $\{(e,e),(f,f),(g,g)\}$. So, all of the following answers are true. Any other answer is WRONG.

$\{(e,e),(f,f),(g,g)\}$

$\{(e,e),(f,f)\}$.

$\{(e,e),(g,g)\}$.

$\{(f,f),(g,g)\}$.

$\{(e,e)\}$.

$\{(f,f)\}$.

$\{(g,g)\}$.

(b) If $(a,b) \in RoR^{-1}$, then $\exists c \in X$, such that $(a,c) \in R^{-1}$ and $(c,b) \in R$. But, $(a,c) \in R^{-1}$ implies that $(c,a) \in R$. And $(c,b) \in R$ implies that $(b,c) \in R^{-1}$. Now since $(a,c) \in R^{-1}$ and $(c,a) \in R$, it follows, by the definition of RoR^{-1}, that $(a,a) \in RoR^{-1}$. Similarly, since $(b,c) \in R^{-1}$ and $(c,b) \in R$, it follows, by the definition of RoR^{-1}, that $(b,b) \in RoR^{-1}$.